RELATIVE DONALDSON-THOMAS THEORY FOR CALABI-YAU 4-FOLDS

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ABSTRACT. Given a complex 4-fold X with an (Calabi-Yau 3-fold) anti-canonical divisor Y, we study relative Donaldson-Thomas invariants for this pair, which are elements in the Donaldson-Thomas cohomologies of Y. We also discuss gluing formulas which relate relative invariants and DT_4 invariants for Calabi-Yau 4-folds.

1. Introduction

Donaldson-Thomas invariants (DT_3 invariants for short) were proposed by Donaldson and Thomas [18], and defined in Thomas' thesis [53]. They count stable sheaves on Calabi-Yau 3-folds, which are related to many other interesting subjects, including Gopakumar-Vafa conjecture on BPS numbers in string theory [23], [25], [32] and MNOP conjecture [42], [43], [44], [49] relating DT_3 invariants to Gromov-Witten invariants. The generalization of DT_3 invariants to count strictly semi-stable sheaves is due to Joyce and Song [31] using Behrend's result [4].

Kontsevich and Soibelman proposed generalized as well as motivic DT theory for Calabi-Yau 3-categories [33], which was later studied by Behrend-Bryan-Szendröi [5] for Hilbert schemes of points. The wall-crossing formula [33], [31] is an important structure for Bridgeland's stability condition [10] and Pandharipande-Thomas invariants [50], [54].

As a categorification of Donaldson-Thomas invariants, Brav, Bussi, Dupont, Joyce and Szendroi [8] and Kiem and Li [32] recently defined a cohomology theory for Calabi-Yau 3-folds whose Euler characteristic is the DT_3 invariant. The point is that moduli spaces of simple sheaves on Calabi-Yau 3-folds are critical points of holomorphic functions locally [9], [31], and we could consider perverse sheaves of vanishing cycles of these functions. They glued these local perverse sheaves and defined its hypercohomology as DT_3 cohomology. In general, such a gluing requires a square root of the determinant line bundle of the moduli space [26], [33], [48].

As an extension of Donaldson-Thomas invariants to Calabi-Yau 4-folds, Borisov and Joyce [7] and the authors [12], [13] developed DT_4 invariants (or 'holomorphic Donaldson invariants') which count stable sheaves on Calabi-Yau 4-folds. It is desirable to construct a TQFT type structure for these DT_4 and DT_3 theories. The purpose of this paper is to make some initial steps in this direction. We remark that Joyce also has a program of establishing TQFT structures on Calabi-Yau 3 and 4-folds [30] using Pantev-Töen-Vaquié-Vezzosi's shifted symplectic structures on derived schemes [51].

Our set-up is a smooth Calabi-Yau 3-fold $Y = s^{-1}(0)$ as an anti-canonical divisor of a complex projective 4-fold X, where $s \in \Gamma(X, K_X^{-1})$. Then $1/s$ is nowhere vanishing inside $X\backslash Y$ which gives a trivialization of its canonical bundle. Thus $X\backslash Y$ is an open Calabi-Yau 4-fold which has a compactification X by adding a compact Calabi-Yau 3-fold.

We consider any Gieseker moduli space $\mathfrak{M}_X$ of semi-stable sheaves which consists of slope-stable bundles only, and assume there exists a restriction morphism $r : \mathfrak{M}_X \to \mathfrak{M}_Y$ to a Gieseker moduli space of stable sheaves on Y (see Theorem 3.1 for its existence). The deformation-obstruction theory associated to r is described as follows: for any stable bundle $E \in \mathfrak{M}_X$, we have an exact sequence

$$0 \to H^1(X, End_0E \otimes K_X) \to H^1(X, End_0E) \to H^1(Y, End_0E|_Y) \to$$

$$\to H^2(X, End_0E \otimes K_X) \to H^2(X, End_0E) \to H^2(Y, End_0E|_Y) \to$$

$$\to H^3(X, End_0E \otimes K_X) \to H^3(X, End_0E) \to 0.$$

Note that the transpose of the above sequence with respect to Serre duality pairings on X and Y is itself [18]. This is the key property for the definition of relative DT_4 virtual cycles, which we define for the following three good cases.

Case I. [Rigid case] If every $E \in \mathfrak{M}_X$ satisfies $H^1(Y, End_0E|_Y) = 0$, then the above long exact sequence breaks into canonical isomorphisms

$$H^1(X, End_0E) \cong H^3(X, End_0E)^*, \quad H^2(X, End_0E) \cong H^2(X, End_0E)^*.$$

Similar to the case of Calabi-Yau 4-folds [7], $\mathfrak{M}_X$ will have a (-2)-shifted symplectic structure in the sense of PTVV [51]. By Borisov-Joyce [7], there exists a virtual cycle, which we define to be the relative DT_4 virtual cycle.

Case II. [Surjective case] If $r : \mathfrak{M}_X \to \mathfrak{M}_Y$ is a surjective map between smooth moduli spaces (throughout this paper, unless specified otherwise, smooth moduli spaces mean their Kuranishi maps are zero), we obtain a canonical isomorphism

$$H^2(X, End_0E) \cong H^2(X, End_0E)^*,$$

which endows the obstruction bundle $Ob_{\mathfrak{M}_X}$ with a non-degenerate quadratic form. Then the relative DT_4 virtual cycle is defined to be the Euler class of the self-dual subbundle of $Ob_{\mathfrak{M}_X}$ as in Definition 5.12 [13].

Case III. [Injective case] If $r : \mathfrak{M}_X \to \mathfrak{M}_Y$ is an injective map between smooth moduli spaces, we obtain an exact sequence

$$0 \to H^1(X, End_0E) \to H^1(Y, End_0E|_Y) \to H^2(X, End_0E \otimes K_X)$$
$$\to H^2(X, End_0E) \to H^2(Y, End_0E|_Y) \to H^3(X, End_0E \otimes K_X) \to 0.$$

This determines a surjective map

$$s : Ob_{\mathfrak{M}_X} \twoheadrightarrow \mathcal{N}^*_{\mathfrak{M}_X/\mathfrak{M}_Y}$$

from the obstruction bundle of $\mathfrak{M}_X$ to the conormal bundle of $\mathfrak{M}_X$ inside $\mathfrak{M}_Y$, and a non-degenerate quadratic form on the reduced bundle $Ob^{red}_{\mathfrak{M}_X} \triangleq Ker(s)$. As in **Case II**, we define the relative DT_4 virtual cycle $[\mathfrak{M}^{rel}_X]^{vir} \in H_*(\mathfrak{M}_X, \mathbb{Z})$ to be the Euler class of the self-dual subbundle of $Ob^{red}_{\mathfrak{M}_X}$. Note that when $\mathfrak{M}_X$ is smooth, r is injective and a neighbourhood of $r(\mathfrak{M}_X) \subseteq \mathfrak{M}_Y$ is smooth, $[\mathfrak{M}^{rel}_X]^{vir}$ can also be defined in a similar way. It is easy to check these definitions of relative DT_4 virtual cycles in **Cases I-III** are all compatible. We compute examples for relative DT_4 virtual cycles in Proposition 3.10.

To sum up, we define the notion of admissibility for Gieseker moduli spaces.

Definition 1.1. Let Y be a smooth anti-canonical divisor of a projective 4-fold X, and $\mathfrak{M}_X$ be a Gieseker moduli space of semi-stable sheaves. $\mathfrak{M}_X$ is admissible with respect to (X, Y) if
(i) $\mathfrak{M}_X$ consists of slope-stable bundles only, and
(ii) there exists a restriction morphism $r : \mathfrak{M}_X \to \mathfrak{M}_Y$ to a Gieseker moduli space of stable sheaves on Y [1].

Theorem/Definition 1.2. *If $\mathfrak{M}_X$ is admissible with respect to (X, Y), then the relative DT_4 virtual cycle exists, i.e.*

$$[\mathfrak{M}^{rel}_X]^{vir} \in H_*(\mathfrak{M}_X, \mathbb{Z}_2)$$

provided that any one of the following conditions holds,
(1) $r(\mathfrak{M}_X)$ is rigid, i.e. $H^1(Y, End_0E|_Y) = 0$ for any $E \in \mathfrak{M}_X$; or
(2) r is surjective between smooth moduli spaces; or
(3) r is injective between smooth moduli spaces (at least when restricted to a neighbourhood of $r(\mathfrak{M}_X)$ in $\mathfrak{M}_Y$).

Furthermore, $[\mathfrak{M}^{rel}_X]^{vir}$ will be defined over integer if $r : \mathfrak{M}_X \to \mathfrak{M}_Y$ has a relative orientation (Definition 1.7) [2].

In general, the virtual dimension of $[\mathfrak{M}^{rel}_X]^{vir}$ is not zero, and we introduce the μ-map to cut it down and define the relative DT_4 invariant. The relative DT_4 invariant is a map

$$v(\mathfrak{M}_X) : Sym^*\big(H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]\big) \to \mathbb{H}^*(\mathfrak{M}_Y, \mathcal{P}^\bullet_{\mathfrak{M}_Y}),$$

where $\mathcal{P}^\bullet_{\mathfrak{M}_Y}$ is the perverse sheaf constructed by Brav-Bussi-Dupont-Joyce-Szendroi [8] and Kiem-Li [32]. In **Cases I-III**, if $\mathfrak{M}_Y$ is smooth, the perverse sheaf $\mathcal{P}^\bullet_{\mathfrak{M}_Y}$ is the $\mathbb{C}$-constant sheaf (up to some degree shift) and the relative DT_4 invariant $v(\mathfrak{M}_X)$ is defined by pairing the relative DT_4 virtual cycle, μ-map and pull-back classes from $H^*(\mathfrak{M}_Y, \mathbb{C})$.

So far, we only work with holomorphic bundles (i.e. $\mathfrak{M}_X$ consists of bundles only). To extend to other coherent sheaves, say ideal sheaves of subschemes, one difficulty is that we will not

[1] Theorem 3.1 ensures we have many such examples.

[2] See Theorem 1.8 and Proposition 1.9 for some partial verification of the existence of relative orientations.

have well-defined restriction maps in the usual sense as subschemes would sit inside the divisor $Y \subseteq X$. To handle this issue, we introduce Li-Wu's good degenerations of Hilbert schemes [39]. Li-Wu's idea is to blow up the divisor once subschemes sit inside Y (like neck-stretching in Donaldson theory) and we are then reduced to consider subschemes which are 'transversal' to Y.

Working with their good degenerations, we study extensions of the above **Cases I-III** to ideal sheaves cases. In particular, we study the obstruction theory of moduli spaces of relative ideal sheaves (Lemma 5.3) and discuss a gluing formula (Theorem 5.12) based on certain conjectures. We also compute examples on relative DT_4 virtual cycles (Example 4.1−Example 4.5) based on Definition 5.4, which include

Example 1.3. (Generic quintic in $\mathbb{P}^4$, Example 4.4)

We take $X = \mathbb{P}^4$ which contains a generic quintic 3-fold $Y = Q$ as its anti-canonical divisor, and consider the primitive curve class $[H] \in H_2(X, \mathbb{Z})$. Ideal sheaves of curves representing this class have Chern character $c = (1, 0, 0, -PD([H]), \frac{3}{2})$ and we denote their moduli space by $I_{\frac{3}{2}}(X, [H])$ ($\cong Gr(2, 5)$). The generic quintic Q contains 2875 rigid degree 1 rational curves and $I_{\frac{3}{2}}(X, [H])$ contains a finite subset S with 2875 points. $I_{\frac{3}{2}}(X, [H])\backslash S$ has a well-defined restriction morphism to $Hilb^5(Q)$. To extend the morphism across those 2875 points, we introduce Li-Wu's expanded pair $X[1]_0 = X \cup \Delta_1$, $Y[1]_0(\cong Q) \subseteq \Delta_1$, where $\Delta_1 \cong \mathbb{P}(\mathcal{O}_Q \oplus \mathcal{O}_{\mathbb{P}^4}(5)|_Q)$, and consider the moduli space $I_{\frac{3}{2}}(X[1]_0, [H])$ of relative ideal sheaves of curves. Geometrically, it is the blow up of $I_{\frac{3}{2}}(X, [H])$ along those 2875 points, i.e.

$$I_{\frac{3}{2}}(X[1]_0, [H]) \cong Bl_S(Gr(2, 5)),$$

where each exceptional divisor corresponds to a $Hilb^5(\mathbb{P}^1)$ for each $\mathbb{P}^1 \subseteq Q$. We then have a injective restriction morphism

$$I_{\frac{3}{2}}(X[1]_0, [H]) \to Hilb^5(Y[1]_0),$$

$$I_C \mapsto I_C|_{Y[1]_0}$$

with smooth image. Conditions in Definition 5.4 (or **Case III**) are satisfied and the relative DT_4 virtual cycle is the usual fundamental class of the moduli space $I_{\frac{3}{2}}(X[1]_0, [H]) \cong Bl_S(Gr(2, 5))$.

Proposition 1.4. *For* $(\mathbb{P}^4, Q)$*, we have a restriction morphism*

$$I_{\frac{3}{2}}(\mathbb{P}^4[1]_0, [H]) \to Hilb^5(Q)$$

from the moduli space of relative ideal sheaves of degree 1 rational curves in $\mathbb{P}^4$ *to the Hilbert scheme of five points in a generic quintic 3-fold. The relative* DT_4 *virtual cycle of* $I_{\frac{3}{2}}(\mathbb{P}^4[1]_0, [H])$ *is the usual fundamental class of the moduli space* $I_{\frac{3}{2}}(\mathbb{P}^4[1]_0, [H]) \cong Bl_S(Gr(2, 5))$.

We adapt Li-Wu's good degenerations to torsion sheaves and verify Conjectures 5.8, 5.9 in the following case.

Example 1.5. (Relative DT_4/DT_3, Example 4.5)

Let $X = Y_1 \times \mathbb{P}^1$ which contains $Y = (Y_1 \times 0) \sqcup (Y_1 \times \infty)$ as an anti-canonical divisor, where Y_1 is a compact Calabi-Yau 3-fold. We denote $\mathfrak{M}_c(Y_1)$ to be a Gieseker moduli space of torsion-free semi-stable sheaves on Y_1 with Chern character $c \in H^{even}(Y_1, \mathbb{Q})$ (we assume there is no strictly semi-stable sheaf), and denote $\mathfrak{M}_c(X)$ to be the moduli space of sheaves on X which are push-forward of stable sheaves in $\mathfrak{M}_c(Y_1 \times t)$ for some t ($\mathfrak{M}_c(X) \cong \mathfrak{M}_c(Y_1) \times \mathbb{P}^1$).

To have a well-defined restriction map, we introduce $X[1]_0 = \Delta_{-1} \cup X \cup \Delta_1$, where $\Delta_{\pm 1} \cong Y_1 \times \mathbb{P}^1$, and consider the relative moduli space $\mathfrak{M}_c(X[1]_0)$ with

$$\mathfrak{M}_c(X[1]_0) \cong \mathfrak{M}_c(Y_1) \times \mathbb{P}^1.$$

Conjectures 5.8, 5.9 hold and the relative DT_4 virtual cycle satisfies

$$[\mathfrak{M}_c^{rel}(X[1]_0)]^{vir} = DT_3(\mathfrak{M}_c(Y_1)) \cdot [\mathbb{P}^1] \in H_2(\mathfrak{M}_c(X[1]_0), \mathbb{Z}),$$

where $DT_3(\mathfrak{M}_c(Y_1))$ is the Donaldson-Thomas invariant defined by Thomas [53].

Theorem 1.6. *For* $(X = Y_1 \times \mathbb{P}^1, Y = Y_1 \times \{0, \infty\})$*, where* Y_1 *is a compact Calabi-Yau 3-fold, we have a restriction map*

$$\mathfrak{M}_c(X[1]_0) \cong \mathfrak{M}_c(Y_1) \times \mathbb{P}^1 \to pt,$$

from the moduli space of relative torsion sheaves coming from push-forward of stable sheaves in $\mathfrak{M}_c(Y_1 \times t)$*, where* $\mathfrak{M}_c(Y_1)$ *is a Gieseker moduli space of torsion-free semi-stable sheaves on* Y_1 *consisting of no strictly semi-stable sheaf.*

The relative DT_4 virtual cycle exists and satisfies

$$[\mathfrak{M}_c^{rel}(X[1]_0)]^{vir} = DT_3(\mathfrak{M}_c(Y_1)) \cdot [\mathbb{P}^1] \in H_2(\mathfrak{M}_c(X[1]_0), \mathbb{Z}),$$

where $DT_3(\mathfrak{M}_c(Y_1))$ is the DT_3 invariant of Y_1 (CY_3) with respect to Chern character c.

Finally, we give a coherent description of the orientability issues involved in **Cases I-III** (Proposition 6.2) and summarize them into the following definition.

Definition 1.7. (Definition 6.1) Let X be a smooth projective 4-fold with a smooth anti-canonical divisor $Y \in |K_X^{-1}|$, and $r : \mathcal{M}_X \to \mathcal{M}_Y$ be a well-defined restriction morphism between coarse moduli spaces of simple sheaves on X and Y with fixed Chern classes respectively. In this case, there exists a canonical isomorphism

$$\alpha : (\mathcal{L}_{\mathcal{M}_X})^{\otimes 2} \cong r^* \mathcal{L}_{\mathcal{M}_Y}.$$

A *relative orientation* for morphism r consists of a square root $(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}$ of the determinant line bundle $\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}}$ and an isomorphism

$$\theta : \mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{red}} \cong r^*(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}$$

such that $\theta \otimes \theta \cong \alpha$ holds over $\mathcal{M}_X^{red}$ for the isomorphism α.

We then give the following partial verification of the existence of orientations.

Theorem 1.8. *(Weak relative orientability, Theorem 6.3)*
Let Y be a smooth anti-canonical divisor in a projective 4-fold X with $Tor(H_*(X,\mathbb{Z})) = 0$, $E \to X$ be a complex vector bundle with structure group $SU(N)$, where $N \gg 0$. Let $\mathcal{M}_X$ be a coarse moduli scheme of simple holomorphic structures on E, which has a well-defined restriction morphism

$$r : \mathcal{M}_X \to \mathcal{M}_Y,$$

to a proper coarse moduli scheme of simple bundles on Y with fixed Chern classes.
Then there exists a square root $(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}$ of $\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}}$ such that

$$c_1(\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{red}}) = r^* c_1((\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}),$$

where $\mathcal{L}_{\mathcal{M}_X}$ (resp. $\mathcal{L}_{\mathcal{M}_Y}$) is the determinant line bundle of $\mathcal{M}_X$ (resp. $\mathcal{M}_Y$).

Another partial verification is given as follows.

Proposition 1.9. *(Proposition 6.4)*
We assume $H^1(\mathcal{M}_X, \mathbb{Z}_2) = 0$. Then relative orientations for restriction morphism $r : \mathcal{M}_X \to \mathcal{M}_Y$ exist.

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2. Review of basic facts in DT theory

2.1. **Some basic facts in DT_4 theory.** We start with a complex projective Calabi-Yau 4-fold $(X, \mathcal{O}_X(1))$ $(Hol(X) = SU(4))$ with a Ricci-flat Kähler metric g [58], a Kähler form ω, a holomorphic four-form Ω, and a topological bundle with a Hermitian metric (E, h). We define

$$*_4 = (\Omega\lrcorner) \circ * : \Omega^{0,2}(X, EndE) \to \Omega^{0,2}(X, EndE),$$

with $*_4^2 = 1$ and it splits the corresponding harmonic subspace into (anti-)self-dual parts.

The DT_4 equations are defined to be

$$\left\{ \begin{array}{l} F_+^{0,2} = 0 \\ F \wedge \omega^3 = 0, \end{array} \right. \tag{1}$$

where the first equation is $F^{0,2} + *_4 F^{0,2} = 0$ and we assume $c_1(E) = 0$ for simplicity in the moment map equation $F \wedge \omega^3 = 0$.

We denote $\mathcal{M}^{DT_4}(X, g, [\omega], c, h)$ or simply $\mathcal{M}_c^{DT_4}$ to be the space of gauge equivalence classes of solutions to the DT_4 equations on E with $ch(E) = c$.

We take $\mathcal{M}_c^{bdl}$ to be the moduli space of slope-stable holomorphic bundles with fixed Chern character c. By the Donaldson-Uhlenbeck-Yau's theorem [55], we can identify it with the moduli space of gauge equivalence classes of solutions to the holomorphic HYM equations

$$\begin{cases} F^{0,2} = 0 \\ F \wedge \omega^3 = 0. \end{cases}$$

By Lemma 4.1 [13], if $ch_2(E) \in H^{2,2}(X, \mathbb{C})$, then $F_+^{0,2} = 0 \Rightarrow F^{0,2} = 0$. In particular, if $\mathcal{M}_c^{bdl} \neq \emptyset$, then $\mathcal{M}_c^{DT_4} \cong \mathcal{M}_c^{bdl}$ as sets.

The comparison of analytic structures is given by

Theorem 2.1. *(Theorem 1.1* [13]*) We assume $\mathcal{M}_c^{bdl} \neq \emptyset$ and fix $d_A \in \mathcal{M}_c^{DT_4}$, then*
(1) there exists a Kuranishi map $\tilde{\kappa}$ of $\mathcal{M}_c^{bdl}$ at $\overline{\partial}_A$ (the (0,1) part of d_A) such that $\tilde{\tilde{\kappa}}_+$ is a Kuranishi map of $\mathcal{M}_c^{DT_4}$ at d_A, where

$$\tilde{\tilde{\kappa}}_+ = \pi_+(\tilde{\tilde{\kappa}}) : H^{0,1}(X, EndE) \xrightarrow{\tilde{\tilde{\kappa}}} H^{0,2}(X, EndE) \xrightarrow{\pi_+} H_+^{0,2}(X, EndE)$$

and π_+ is projection to the self-dual forms;
(2) the closed imbedding between analytic spaces possibly with non-reduced structures $\mathcal{M}_c^{bdl} \hookrightarrow \mathcal{M}_c^{DT_4}$ is also a homeomorphism between topological spaces.

Remark 2.2. By Proposition 10.10 [13], the map $\tilde{\tilde{\kappa}}$ satisfies $Q_{Serre}(\tilde{\tilde{\kappa}}, \tilde{\tilde{\kappa}}) \geq 0$, where Q_{Serre} is the Serre duality pairing on $H^{0,2}(X, EndE)$.

To define Donaldson type invariants using $\mathcal{M}_c^{DT_4}$, we need to give it a good compactification (such that it carries a deformation invariant fundamental class). For this purpose, we introduce $\mathcal{M}_c(X, \mathcal{O}_X(1))$ or simply $\mathcal{M}_c$ to be the Gieseker moduli space of $\mathcal{O}_X(1)$-stable sheaves on X with given Chern character c. Motivated by Theorem 2.1, we make the following definition.

Definition 2.3. ([13]) We call a C^∞-scheme, $\overline{\mathcal{M}}_c^{DT_4}$ generalized DT_4 moduli space if there exists a homeomorphism

$$\mathcal{M}_c \to \overline{\mathcal{M}}_c^{DT_4}$$

such that at each closed point of $\mathcal{M}_c$, say $\mathcal{F}$, $\overline{\mathcal{M}}_c^{DT_4}$ is locally isomorphic to $\kappa_+^{-1}(0)$, where

$$\kappa_+ = \pi_+ \circ \kappa : Ext^1(\mathcal{F}, \mathcal{F}) \to Ext_+^2(\mathcal{F}, \mathcal{F}),$$

κ is a Kuranishi map at $\mathcal{F}$ and $Ext_+^2(\mathcal{F}, \mathcal{F})$ is a half dimensional real subspace of $Ext^2(\mathcal{F}, \mathcal{F})$ on which the Serre duality quadratic form is real and positive definite.

Remark 2.4.
1. The existence of generalized DT_4 moduli spaces was proved by Borisov-Joyce [7]. The authors proved their existence as real analytic spaces in certain cases and defined the corresponding virtual fundamental classes [12],[13].
2. For fixed data $(X, \mathcal{O}_X(1), c)$, generalized DT_4 moduli spaces may not be unique. However, they all carry the same virtual fundamental class [7].

The proof of Borisov-Joyce's gluing result is divided into two parts. Firstly, they use good local models of $\mathcal{M}_c$, i.e. local 'Darboux charts' in the sense of Brav, Bussi and Joyce [9]. Then they chose the half dimensional real subspace $Ext_+^2(\mathcal{F}, \mathcal{F})$ appropriately and use partition of unity and homotopical algebra to glue κ_+. We state the analytic version of BBJ's local Darboux charts as follows.

Theorem 2.5. *(Brav-Bussi-Joyce* [9] *Corollary 5.20, see also Theorem 10.7* [13]*)*
Let $\mathcal{M}_c$ be a Gieseker moduli space of stable sheaves on a compact Calabi-Yau 4-fold X. Then for any closed point $\mathcal{F} \in \mathcal{M}_c$, there exists an analytic neighborhood $U_{\mathcal{F}} \subseteq \mathcal{M}_c$, a holomorphic map near the origin

$$\kappa : Ext^1(\mathcal{F}, \mathcal{F}) \to Ext^2(\mathcal{F}, \mathcal{F})$$

such that $Q_{Serre}(\kappa, \kappa) = 0$ and $\kappa^{-1}(0) \cong U_{\mathcal{F}}$ as complex analytic spaces possibly with non-reduced structures, where Q_{Serre} is the Serre duality pairing on $Ext^2(\mathcal{F}, \mathcal{F})$.

Proof. (See the Proof of Theorem 10.7 [13]) The point is that we can use Seidel-Thomas twists [31],[52] transfer the problem to a problem on moduli spaces of holomorphic bundles and then notice $ch_2(E) \wedge \Omega = 0$ for holomorphic bundle E, where Ω is the holomorphic top form. $\square$

To make sense of virtual fundamental classes of generalized DT_4 moduli spaces as homology classes in $\mathcal{M}_c$'s, one would in general rely on Joyce's D-manifolds theory [29] or Fukaya-Oh-Ohta-Ono's theory of Kuranishi spaces [21] or Hofer's polyfolds theory [24] (see [57] for a comparison between them). Assuming this part, which is claimed by Borisov-Joyce, we have

Theorem 2.6. *(Borisov-Joyce* [7], [30]*)*
Let X be a complex projective Calabi-Yau 4-fold, and $\mathcal{M}_c$ be a Gieseker moduli space of stable sheaves which is compact (it is true if the degree and rank of sheaves are coprime). Then there exists a generalized DT_4 moduli space $\overline{\mathcal{M}}_c^{DT_4}$. If we further assume $\overline{\mathcal{M}}_c^{DT_4}$ is orientable in the sense of D-manifold, then the virtual fundamental class of $\overline{\mathcal{M}}_c^{DT_4}$ exists and is a well-defined homology class, i.e.

$$[\overline{\mathcal{M}}_c^{DT_4}]^{vir} \in H_*(\mathcal{M}_c, \mathbb{Z}),$$

which coincides with earlier definitions of DT_4 virtual cycles (Definition 5.3, 5.12, 5.14 [13]*).*

We can furthermore define the DT_4 invariant by pairing the above cycle with μ-map as in Definition 5.15 [13]*. With appropriate choice of orientations, DT_4 invariants are invariant under deformations of complex structures of X.*

Because of the Serre duality for $Ext^*(\mathcal{F},\mathcal{F})$, the existence of an orientation on a generalized DT_4 moduli space $\overline{\mathcal{M}}_c^{DT_4}$ (in the sense of D-manifold) is equivalent to the existence of a reduction of the structure group of $(\mathcal{L}_X, Q_{Serre})$ to $SO(1,\mathbb{C})$, where $\mathcal{L}_X$ is the determinant line bundle with $\mathcal{L}_X|_{\mathcal{F}} \cong (\wedge^{top} Ext^{even}(\mathcal{F},\mathcal{F}))^{-1} \otimes \wedge^{top} Ext^{odd}(\mathcal{F},\mathcal{F})$ and Q_{Serre} is the Serre duality quadratic form on it.

Theorem 2.7. *(Theorem 2.2* [14]*) Let X be a compact Calabi-Yau 4-fold with $H_{odd}(X,\mathbb{Z}) = 0$. For any Gieseker moduli space $\mathcal{M}_c$ of stable sheaves, the structure group of $(\mathcal{L}_X, Q_{Serre})$ can be reduced to $SO(1,\mathbb{C})$.*

In the case when the Gieseker moduli space $\mathcal{M}_c$ of stable sheaves on X is smooth (i.e. Kuranishi maps are zero), the obstruction sheaf Ob such that $Ob|_{\mathcal{F}} \cong Ext^2(\mathcal{F},\mathcal{F})$ is a bundle with Serre duality quadratic form. There exists a real subbundle Ob_+ with positive definite quadratic form such that $Ob \cong Ob_+ \otimes_{\mathbb{R}} \mathbb{C}$ as vector bundles with quadratic forms. We call Ob_+ the self-dual obstruction bundle. By Definition 5.12 [13], the virtual fundamental class of $\overline{\mathcal{M}}_c^{DT_4}$ is the Poincaré dual of the Euler class of the self-dual obstruction bundle (if it is orientable), i.e.

$$[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = PD(e(Ob_+)) \in H_*(\mathcal{M}_c, \mathbb{Z}). \tag{2}$$

This motivates later definitions of relative DT_4 invariants.

2.2. **Some basic facts in DT_3 theory.** Moduli spaces of simple sheaves on CY_3's are locally critical points of holomorphic functions [9], [31] and we can consider the perverse sheaves of vanishing cycles of these functions. The expected cohomology which categorifies DT_3 invariant is defined by first gluing these local perverse sheaves and then taking its hypercohomology.

Theorem 2.8. *(Brav-Bussi-Dupont-Joyce-Szendroi* [8]*, Kiem-Li* [32]*)*
Let Y be a Calabi-Yau 3-fold over $\mathbb{C}$, and $\mathcal{M}$ a classical moduli scheme of simple coherent sheaves or simple complexes of coherent sheaves on Y, with natural (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}^\bullet_{\mathcal{M}}$. Suppose we are given a square root of $det(\mathcal{E}^\bullet)$, then there exists a perverse sheave $\mathcal{P}^\bullet_{\mathcal{M}}$ uniquely up to canonical isomorphism such that the Euler characteristic of its hypercohomology is the Donaldson-Thomas invariant [53]*.*

2.3. **An overview of TQFT type structures in DT_4-DT_3 theories.** In this subsection, we give an overview of TQFT structures in gauge theories on Calabi-Yau 3-folds and 4-folds.

We take a smooth (Calabi-Yau) 3-fold Y in a complex projective 4-fold X^+ as its anti-canonical divisor, and consider a moduli space $\mathfrak{M}_{X^+}$ of stable bundles with fixed Chern classes on X^+ which has a well-defined restriction morphism

$$r : \mathfrak{M}_{X^+} \to \mathfrak{M}_Y$$

to a moduli space of stable sheaves on Y. This would determine a class $v(\mathfrak{M}_{X^+}) \in \mathbb{H}^*(\mathcal{P}^\bullet_{\mathfrak{M}_Y})$.

Given another complex projective 4-fold X^- which contains Y as its anti-canonical divisor, we form a singular space $X_0 = X^+ \cup_Y X^-$. When X_0 admits a smooth deformation X_t, X_t will be a family of CY_4's provided that the normal bundle of Y in $X^\pm$ is trivial. As the perverse sheaf in Theorem 2.8 is self-dual under the Verdier duality, i.e. $\mathbb{D}_{\mathfrak{M}_Y}(\mathcal{P}^\bullet_{\mathfrak{M}_Y}) \cong \mathcal{P}^\bullet_{\mathfrak{M}_Y}$ [8], we can define $\langle v(\mathfrak{M}_{X^+}), v(\mathfrak{M}_{X^-})\rangle$ using the Verdier duality on $\mathbb{H}^*(\mathfrak{M}_Y, \mathcal{P}^\bullet_{\mathfrak{M}_Y})$ as long as $\mathfrak{M}_Y$ is compact.

Ignoring the stability issue and contributions from general coherent sheaves, we ask the following question which can be regarded as a *complexification* of Chern-Simons-Donaldson-Floer TQFT structure for 3 and 4-manifolds (see Atiyah [1]).

Question 2.9. What is the relation between DT_4 invariants and relative DT_4 invariants, namely, comparing $DT_4(\mathfrak{M}_{X_t})$ with $\langle v(\mathfrak{M}_{X^+}), v(\mathfrak{M}_{X^-})\rangle$?

3. Relative DT_4 invariants for holomorphic bundles

3.1. **Definitions of relative DT_4 invariants.** In this section, we restrict to some good cases and define rigorously the relative DT_4 invariant mentioned before, i.e. $v(\mathfrak{M}_{X^+}) \in \mathbb{H}^*(\mathcal{P}^{\bullet}_{\mathfrak{M}_Y})$. To have a well-defined restriction map, in this section, we assume all Gieseker moduli spaces of semi-stable sheaves on 4-folds consist of slope-stable bundles only. This will serve as a model for the later study of relative DT_4 invariants for ideal sheaves.

Let Y be a smooth anti-canonical divisor of a smooth projective 4-fold X, $\mathfrak{M}_X$ be a Gieseker moduli space which is admissible with respect to (X,Y) (see Definition 1.1), and

$$r : \mathfrak{M}_X \to \mathfrak{M}_Y$$

be the restriction morphism to a Gieseker moduli space on Y. We recall the following criterion which ensures that we have such morphism r in many cases.

Theorem 3.1. *(Flenner* [20]*)*
Let $(X,\mathcal{O}_X(1))$ be a complex n-dimensional normal projective variety with $\mathcal{O}_X(1)$ very ample. We take $\mathcal{F}$ to be a $\mathcal{O}_X(1)$-slope semi-stable torsion-free sheaf of rank r. d and $1 \leq c \leq n-1$ are integers such that

$$[\begin{pmatrix} n+d \\ d \end{pmatrix} - cd - 1]/d > \deg(\mathcal{O}_X(1)) \cdot \max(\frac{r^2-1}{4}, 1).$$

Then for a generic complete intersection $Y = H_1 \cap \cdots \cap H_c$ with $H_i \in |\mathcal{O}_X(d)|$, $\mathcal{F}|_Y \triangleq \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is $\mathcal{O}_X(1)|_Y$-slope semi-stable on Y.

Remark 3.2. For $X = \mathbb{P}^4$, $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^4}(1)$ is very ample. We take $c = 1$ and $d = 5$, then any rank $r \leq 9$ semi-stable sheaf on $\mathbb{P}^4$ remains semi-stable when restricted to a generic quintic 3-fold inside.

The deformation-obstruction theory associated to the restriction morphism r is described by the following exact sequence.

Lemma 3.3. *We take a stable bundle $E \in \mathfrak{M}_X$, and assume Y is connected, then we have a long exact sequence,*

$$\begin{aligned} 0 &\to H^1(X, EndE \otimes K_X) \to H^1(X, EndE) \to H^1(Y, EndE|_Y) \to \\ &\to H^2(X, EndE \otimes K_X) \to H^2(X, EndE) \to H^2(Y, EndE|_Y) \to \\ &\to H^3(X, EndE \otimes K_X) \to H^3(X, EndE) \to 0. \end{aligned}$$

Proof. We tensor $0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ with $EndE$ and take its cohomology. □

We note that the transpose of the above sequence with respect to Serre duality pairing on X and Y remains the same (see also [18]). This will be the key property used in the following definitions of relative DT_4 invariants.

Case I: when $\mathfrak{M}_Y$ is of expected dim. If we assume $H^1(Y, EndE|_Y) = 0$ for any $E \in \mathfrak{M}_X$, then $H^2(Y, EndE|_Y) = 0$ and $\mathfrak{M}_Y$ contains components of finite points, labeled by $E_1, ..., E_m$ which come from restrictions of bundles on X. We denote $\mathfrak{M}_{X,E_i}$ to be components of $\mathfrak{M}_X$ such that $r(\mathfrak{M}_{X,E_i}) = E_i$. By Lemma 3.3, we have canonical isomorphisms

$$H^3(X, EndE)^* \cong H^1(X, EndE), \quad H^2(X, EndE)^* \cong H^2(X, EndE).$$

In fact, we apply Theorem 2.13 of [11] and take the induced shifted symplectic structure on the stable loci as in [7], $\mathfrak{M}_{X,E_i}$ has a (-2)-shifted symplectic structure in the sense of [51]. Analogs to Theorem 2.6, we can define the relative DT_4 virtual cycle $[\mathfrak{M}^{rel}_{X,E_i}]^{vir} \in H_n(\mathfrak{M}_{X,E_i}, \mathbb{Z}_2)$ with $n = 1 - \chi(X, EndE)$. The cycle will be defined over integer if the associated D-manifold of $\mathfrak{M}_{X,E_i}$ is orientable.

The virtual dimension is not zero in general, we introduce a μ-map as in Definition 5.15 [13].

Definition 3.4. We denote the universal sheaf of $\mathfrak{M}_X$ by $\mathfrak{F} \to \mathfrak{M}_X \times X$.

The μ-map is

$$\mu : H_*(X) \otimes \mathbb{Z}[x_1, x_2, ...] \to H^*(\mathfrak{M}_X),$$
$$\mu(\gamma, P) = P(c_1(\mathfrak{F}), c_2(\mathfrak{F}), ...)/\gamma.$$

Pairing virtual cycles with μ-maps defines the relative invariants.

Definition 3.5. Let $\mathfrak{M}_X$ be a Gieseker moduli space of semi-stable sheaves which is admissible with respect to (X, Y) (see Definition 1.1), and $r : \mathfrak{M}_X \to \mathfrak{M}_Y$ be the restriction morphism.

We assume $H^1(Y, EndE|_Y) = 0$ for any $E \in \mathfrak{M}_X$, then the relative DT_4 invariant is a map

$$v(\mathfrak{M}_X) : Sym^*\big(H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]\big) \to \mathbb{H}^*(\mathfrak{M}_Y, \mathcal{P}^\bullet_{\mathfrak{M}_Y}) \tag{3}$$

such that

$$v(\mathfrak{M}_X)((\gamma_1, P_1), (\gamma_2, P_2), ...) = \sum_{i=1}^{m} \langle [\mathfrak{M}^{rel}_{X,E_i}]^{vir}, \mu(\gamma_1, P_1) \cup \mu(\gamma_2, P_2) \cup ..., \rangle E_i,$$

where $\langle , \rangle$ denotes the natural pairing between homology and cohomology classes and $\{E_i\}_{1 \leq i \leq m}$ are taken as a basis of $H^*(r(\mathfrak{M}_X))$.

Case II: when $\mathfrak{M}_X$ and $\mathfrak{M}_Y$ are smooth and r is surjective. We assume $\mathfrak{M}_X$ and $\mathfrak{M}_Y$ are smooth (i.e. all Kuranishi maps are zero) and the restriction map r is surjective. By Lemma 3.3, we get a canonical isomorphism

$$H^2(X, EndE)^* \cong H^2(X, EndE)$$

which endows $H^2(X, EndE)$ a non-degenerate quadratic form, and a short exact sequence

$$0 \to H^3(X, EndE)^* \to H^1(X, EndE) \to H^1(Y, EndE|_Y) \to 0.$$

Counting dimensions, we have

$$2h^1(X, EndE) - h^2(X, EndE) = h^1(Y, EndE|_Y) - \chi(X, EndE) + 1,$$

which is a constant on components of $\mathfrak{M}_X$ by assumptions. Similar to (2), the self-dual subbundle of the obstruction bundle $Ob_{\mathfrak{M}_X}$ exists. If it is also orientable, we define the relative DT_4 virtual cycle $[\mathfrak{M}^{rel}_X]^{vir} \in H_n(\mathfrak{M}_X, \mathbb{Z})$ to be the Euler class of the self-dual obstruction bundle, where $n = h^1(Y, EndE|_Y) - \chi(X, EndE) + 1$.

Definition 3.6. Let $\mathfrak{M}_X$ be a Gieseker moduli space of semi-stable sheaves which is admissible with respect to (X, Y) (see Definition 1.1), and $r : \mathfrak{M}_X \to \mathfrak{M}_Y$ be the restriction morphism.

We assume r is surjective between smooth moduli spaces, then the relative DT_4 invariant is a map

$$v(\mathfrak{M}_X) : Sym^*\big(H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]\big) \to H^*(\mathfrak{M}_Y) \tag{4}$$

such that

$$v(\mathfrak{M}_X)((\gamma_1, P_1), (\gamma_2, P_2), ...)(\alpha) = \langle [\mathfrak{M}^{rel}_X]^{vir}, (r^*\alpha) \cup \mu(\gamma_1, P_1) \cup \mu(\gamma_2, P_2) \cup ..., \rangle,$$

where $\alpha \in H^*(\mathfrak{M}_Y)$ and we identify $H^*(\mathfrak{M}_Y) \cong H^*(\mathfrak{M}_Y)^*$ via Poincaré pairing, $\langle , \rangle$ denotes the natural pairing between homology and cohomology classes.

Case III: when $\mathfrak{M}_X$ and $\mathfrak{M}_Y$ are smooth and r is injective. We assume $\mathfrak{M}_X$ and $\mathfrak{M}_Y$ are smooth (i.e. all Kuranishi maps are zero) and the restriction map r is injective. By Lemma 3.3, we get $H^3(X, EndE) = 0$ and an exact sequence

$$0 \to H^1(X, EndE) \to H^1(Y, EndE|_Y) \to H^2(X, EndE)^* \to$$
$$\to H^2(X, EndE) \to H^1(Y, EndE|_Y)^* \to H^1(X, EndE)^* \to 0.$$

This determine a surjective map

$$s : Ob_{\mathfrak{M}_X} \twoheadrightarrow \mathcal{N}^*_{\mathfrak{M}_X/\mathfrak{M}_Y}$$

and a non-degenerate quadratic form on the reduced bundle $Ob^{red}_{\mathfrak{M}_X} \triangleq Ker(s)$, where $Ob_{\mathfrak{M}_X}$ is the obstruction bundle of $\mathfrak{M}_X$ with $Ob_{\mathfrak{M}_X}|_E = H^2(X, EndE)$ and $\mathcal{N}^*_{\mathfrak{M}_X/\mathfrak{M}_Y}$ is the conormal bundle of $\mathfrak{M}_X$ inside $\mathfrak{M}_Y$.

Then if the self-dual subbundle of $Ob^{red}_{\mathfrak{M}_X}$ is orientable, we define the relative DT_4 virtual cycle $[\mathfrak{M}^{rel}_X]^{vir} \in H_n(\mathfrak{M}_X, \mathbb{Z})$ to be the Euler class of it, where the virtual dimension is $n = 2h^1(X, EndE) - \big(h^2(X, EndE) - codim_{\mathfrak{M}_Y}(\mathfrak{M}_X)\big) = h^1(Y, EndE|_Y) - \chi(X, EndE) + 1$. Note

that when $\mathfrak{M}_X$ is smooth, r is injective and a neighbourhood of $r(\mathfrak{M}_X) \subseteq \mathfrak{M}_Y$ is smooth, we could also define $[\mathfrak{M}_X^{rel}]^{vir}$ in a similar way.

Definition 3.7. Let $\mathfrak{M}_X$ be a Gieseker moduli space of semi-stable sheaves which is admissible with respect to (X, Y) (see Definition 1.1), and $r : \mathfrak{M}_X \to \mathfrak{M}_Y$ be the restriction morphism.

We assume r is injective between smooth moduli spaces (at lease when restricted to a neighbourhood of $r(\mathfrak{M}_X)$ in $\mathfrak{M}_Y$), then the relative DT_4 invariant is a map

$$v(\mathfrak{M}_X) : Sym^*\big(H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]\big) \to H^*(\mathfrak{M}_Y) \tag{5}$$

such that

$$v(\mathfrak{M}_X)((\gamma_1, P_1), (\gamma_2, P_2), ...)(\alpha) = \langle [\mathfrak{M}_X^{rel}]^{vir}, (r^*\alpha) \cup \mu(\gamma_1, P_1) \cup \mu(\gamma_2, P_2) \cup ..., \rangle,$$

where $\alpha \in H^*(\mathfrak{M}_Y)$ and we identify $H^*(\mathfrak{M}_Y) \cong H^*(\mathfrak{M}_Y)^*$ via Poincaré pairing, $\langle , \rangle$ denotes the natural pairing between homology and cohomology classes.

Remark 3.8. It is easy to check Definition 3.5, 3.6 and 3.7 are all compatible.

In general, one could consider moduli spaces of complexes of simple sheaves [41] on a complex projective 4-fold X and resolve complexes of sheaves by complexes of holomorphic bundles, then there will be a natural restriction morphism to a moduli of simple bundles on an anti-canonical divisor of X. A similar long exact sequence in Lemma 3.3 still works and we could study virtual cycle constructions as in **Cases I-III**.

Endomorphisms of DT_3 cohomologies from relative DT_4 invariants. By considering the trace-free version of Lemma 3.3, the above definitions extend to any disconnected divisor $Y \subseteq X$. We are particularly interested in the case when $X = Y_1 \times \mathbb{P}^1$, where Y_1 is a compact Calabi-Yau 3-fold. Then $Y = (Y_1 \times 0) \sqcup (Y_1 \times \infty)$ will be a smooth anti-canonical divisor of X and the relative DT_4 invariant in general is a map

$$v(\mathfrak{M}_X) : Sym^*\big(H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]\big) \to \mathbb{H}^*(\mathfrak{M}_Y, \mathcal{P}^\bullet_{\mathfrak{M}_Y}) \cong \mathbb{H}^*(\mathfrak{M}_{Y_1}, \mathcal{P}^\bullet_{\mathfrak{M}_{Y_1}}) \otimes \mathbb{H}^*(\mathfrak{M}_{Y_1}, \mathcal{P}^\bullet_{\mathfrak{M}_{Y_1}}).$$

By the Verdier duality and $\mathbb{D}_{\mathfrak{M}_{Y_1}}(\mathcal{P}^\bullet_{\mathfrak{M}_{Y_1}}) \cong \mathcal{P}^\bullet_{\mathfrak{M}_{Y_1}}$, we have

$$\mathbb{H}^*(\mathfrak{M}_{Y_1}, \mathcal{P}^\bullet_{\mathfrak{M}_{Y_1}}) \cong \mathbb{H}^*(\mathfrak{M}_{Y_1}, \mathcal{P}^\bullet_{\mathfrak{M}_{Y_1}})^*.$$

Thus a relative DT_4 invariant of $(Y_1 \times \mathbb{P}^1, Y_1 \times \{0, \infty\})$ determines some endomorphisms of the DT_3 cohomology

$$v(\mathfrak{M}_X) : Sym^*\big(H_*(Y_1)[t]/(t^2) \otimes \mathbb{Z}[x_1, x_2, ...]\big) \to End_{\mathbb{C}}\big(\mathbb{H}^*(\mathfrak{M}_{Y_1}, \mathcal{P}^\bullet_{\mathfrak{M}_{Y_1}})\big)$$

for any Calabi-Yau 3-fold Y_1. In general, the above endomorphisms should be used to establish the gluing formula mentioned in Question 2.9 (see [34] for the real 4-3 dimensional picture).

3.2. **Li-Qin's examples.** We have Li-Qin's examples when conditions in **Case II, III** are satisfied [40]. Let Y be a generic smooth hyperplane section in $X = \mathbb{P}^1 \times \mathbb{P}^3$ of bi-degree $(2, 4)$,

$$c = [1 + (-1, 1)] \cdot [1 + (\epsilon_1 + 1, \epsilon_2 - 1)],$$

$$c|_Y = [1 + (-1, 1)|_Y] \cdot [1 + (\epsilon_1 + 1, \epsilon_2 - 1)|_Y],$$

$$k = (1 + \epsilon_1) \begin{pmatrix} 5 - \epsilon_2 \\ 3 \end{pmatrix}, \quad \epsilon_1, \epsilon_2 = 0, 1, \quad L_r = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, r).$$

We denote $\mathfrak{M}_c(L_r)$ to be the moduli space of L_r-slope stable rank-2 bundles on X with a Chern class c and $\overline{\mathfrak{M}}_{c|_Y}(L_r|_Y)$ to be the moduli space of Gieseker $L_r|_Y$-semistable rank-2 torsion-free sheaves on Y with Chern class $c|_Y$.

Lemma 3.9. *(Li-Qin [40])*
(i) If

$$\frac{4(2 - \epsilon_2)}{2 + 2\epsilon_1 + \epsilon_2} < r < \frac{4(2 - \epsilon_2)}{\epsilon_1 \epsilon_2},$$

then $\overline{\mathfrak{M}}_{c|_Y}(L_r|_Y) \cong \mathbb{P}^k$ and consists of rank-2 stable bundles. Furthermore, the restriction map

$$r : \mathfrak{M}_c(L_r) \to \overline{\mathfrak{M}}_{c|_Y}(L_r|_Y)$$

is well-defined and an isomorphic between projective varieties.

(ii) If $0 < r < \frac{4(2-\epsilon_2)}{2+2\epsilon_1+\epsilon_2}$, then $\mathfrak{M}_c(L_r)$ and $\overline{\mathfrak{M}}_{c|_Y}(L_r|_Y)$ are empty.

Proposition 3.10. *In the above example, for any stable bundle $E \in \mathfrak{M}_c(L_r)$ on $X = \mathbb{P}^1 \times \mathbb{P}^3$,*

$$Ext^1_X(E,E) \cong Ext^1_Y(E|_Y, E|_Y) \cong Ext^2_Y(E|_Y, E|_Y)^* \cong \mathbb{C}^k,$$

$$Ext^i_X(E,E) = 0, \quad \textit{if } i \geq 2.$$

The relative DT_4 virtual cycle $[\mathfrak{M}^{rel}_c(L_r)]^{vir} = [\mathfrak{M}_c(L_r)] \in H_{2k}(\mathfrak{M}_c(L_r), \mathbb{Z})$.

4. Computational examples of relative DT_4 invariants for ideal sheaves

In the above section, we studied relative DT_4 invariants for holomorphic bundles. To formulate the gluing formula, we need to have a good understanding of how stable sheaves could be degenerated into union of stable sheaves on irreducible components of degenerated varieties. At this moment, we will restrict ourselves to the ideal sheaves case where degenerations have simpler behavior.

We take a simple degeneration $\pi : \mathcal{X} \to C$ of projective manifolds over a pointed smooth curve $(C, 0 \in C)$, i.e. (1) $\mathcal{X}$ is smooth, π is projective and smooth away from the central fiber $X_0 = \pi^{-1}(0)$, (2) X_0 is a union of two smooth irreducible components X_+, X_- intersecting transversally along a smooth divisor Y. When generic fibers X_t's are Calabi-Yau 4-folds and Y is an anti-canonical divisor of X_+, X_-, we will study relative DT_4 invariants of ideal sheaves for pairs $(X_\pm, Y)$ and discuss their relations with DT_4 invariants of X_t, $t \neq 0$. The basic technique is the degeneration method developed by J. Li and B. Wu [36], [37], [39], [56].

Li-Wu's construction will be recalled in the appendix and the associated obstruction theory is studied there. In this section, we concentrate on computational examples of relative DT_4 virtual cycles for ideal sheaves based on the extension (Definition 5.4) of constructions for bundles (**Cases I-III**).

Example 4.1. (Ideal sheaves of one point)

We take a compact simply connected 4-fold[1] X_+ which contains a smooth Calabi-Yau 3-fold Y as its anti-canonical divisor. We consider the moduli space $I_1(X_+, 0)$ of structure sheaves of one point (it is equivalent to consider ideal sheaves of one point) which has a well-defined restriction map to Y if points sit inside $X_+\backslash Y$.

To extend the map to the whole moduli space, we introduce Li-Wu's expanded pair, i.e. we consider $X_+[1]_0 = X_+ \cup_Y \Delta$, $Y[1]_0 \subseteq \Delta$, where $\Delta \cong \mathbb{P}(\mathcal{O}_Y \oplus \mathcal{N}_{Y/X_+})$ and form the moduli space $I_1(X_+[1]_0, 0)$ $(\cong X_+)$ of relative structure sheaf of one point, which is the union of $X_+\backslash Y$ with the $\mathbb{C}^*$-equivalence classes of points in $\Delta\backslash((Y \times 0) \cup (Y \times \infty))$. By the Koszul resolution and Serre duality, we have canonical isomorphism

$$Ext^*(\mathcal{O}_P, \mathcal{O}_P) \cong \wedge^*(TX_+|_P).$$

Then the obstruction bundle $Ob = \wedge^2 TX_+$ has a non-degenerate quadratic form only when it is restricted to $X_+\backslash Y$ and $\Delta\backslash((Y\times 0)\cup(Y\times\infty))$, but they do not glue to become a non-degenerate quadratic form on $Ob \to X_+$ as $c_1(Ob) \neq 0$.

If we assume K_{X_+} has a square root $K^{\frac{1}{2}}_{X_+}$ (see the appendix), and form $\widetilde{Ob} \triangleq \wedge^2 TX_+ \otimes K^{\frac{1}{2}}_{X_+}$, then there exists a non-degenerate quadratic form

$$(\wedge^2 TX_+ \otimes K^{\frac{1}{2}}_{X_+}) \otimes (\wedge^2 TX_+ \otimes K^{\frac{1}{2}}_{X_+}) \to \mathcal{O}_{X_+}.$$

As $\pi_1(X) = 0$, the self-dual obstruction bundle $\widetilde{Ob}_+$ is orientable, and Conjectures 5.8, 5.9 hold for this case. To calculate the Euler class $e(\widetilde{Ob}_+)$, we consider the case when $X_+ = Y_1 \times \mathbb{P}^1$ and $Y = (Y_1 \times 0) \sqcup (Y_1 \times \infty)$, where Y_1 is a smooth Calabi-Yau 3-fold. Then

$$\wedge^2 TX_+ \otimes K^{\frac{1}{2}}_{X_+} \cong (\wedge^2 TY_1 \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1)) \oplus (TY_1 \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)),$$

where both factors are maximal isotropic subbundles of $\widetilde{Ob}_+$. By [19] or Lemma 5.13 [13],

$$e(\widetilde{Ob}_+) = \pm\big(c_3(Y_1) + c_2(Y_1) \cdot c_1(\mathcal{O}_{\mathbb{P}^1}(1))\big) = \pm\big(c_3(X_+) - \frac{1}{2} c_1(X_+) \cdot c_2(X_+)\big),$$

where the sign depends on the orientation of $\widetilde{Ob}_+$. The relative DT_4 virtual cycle for structure sheaves of one point is the Poincaré dual of $e(\widetilde{Ob}_+)$ (2).

We then consider examples of ideal sheaves of curves.

[1] Note that any compact Fano manifold is simply connected.

Example 4.2. Let $Q \subseteq \mathbb{P}^4$ be a smooth generic quintic 3-fold. We take $X_+ = Q \times \mathbb{P}^1$ with an anti-canonical divisor $Y = (Q \times 0) \sqcup (Q \times \infty)$. Then $H_2(X_+, \mathbb{Z}) \cong H_2(Q, \mathbb{Z}) \oplus H_2(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

(i) We first fix the curve class to be the generator $[H] \in H_2(\mathbb{P}^1, \mathbb{Z}) \subseteq H_2(X_+, \mathbb{Z})$. The moduli space $I_0(X_+, [H])$ $(\cong Q)$ of ideal sheaves of curves with Chern character $c = (1, 0, 0, -PD([H]), 0)$ consists of ideal sheaves of curves of type $\{pt\} \times \mathbb{P}^1$, $pt \in Q$. We have a restriction morphism

$$r : I_0(X_+, [H]) \to I_1(Q, 0) \times I_1(Q, 0) \cong Q \times Q,$$

$$I_{\{pt\} \times \mathbb{P}^1} \mapsto (I_{pt}, I_{pt}),$$

to the moduli space of ideal sheaves of one point in Y. r is the diagonal embedding if we identify $I_0(X_+, [H]) \cong Q$. By direct calculations, for any $I_C \in I_0(X_+, [H])$, we have

$$Ext^i_{X_+}(I_C, I_C) \cong \mathbb{C}^3, \ i = 1, 2, \quad Ext^3_{X_+}(I_C, I_C) \cong \mathbb{C}, \quad Ext^4_{X_+}(I_C, I_C) = 0.$$

Analogs to Lemma 5.3, we have a long exact sequence,

$$0 \to Ext^1_{X_+}(I_C, I_C) \to Ext^1_Y(I_C|_Y, I_C|_Y) \to Ext^2_{X_+}(I_C, I_C \otimes K_{X_+})$$

$$\to Ext^2_{X_+}(I_C, I_C) \to Ext^2_Y(I_C|_Y, I_C|_Y) \to Ext^3_{X_+}(I_C, I_C \otimes K_{X_+}) \to 0.$$

This determines a surjective morphism

$$s : Ob \twoheadrightarrow \mathcal{N}_{Q/Q \times Q},$$

from the obstruction bundle Ob with $Ob|_{I_C} = Ext^2_{X_+}(I_C, I_C)$ to the conormal bundle of $I_0(X_+, [H])$ in $I_1(Q, 0)$. Furthermore, $rk(Ob) = codim(Q, Q \times Q) = 3$ and conditions in Definition 5.4 are satisfied. The relative DT_4 virtual cycle is the usual fundamental class of the moduli space, i.e. $[I_0(X_+, [H])] \in H_6(I_0(X_+, [H]), \mathbb{Z})$.

(ii) If we fix the curve class to be the generator $[H_Q] \in H_2(Q, \mathbb{Z}) \subseteq H_2(X_+, \mathbb{Z})$, the moduli space $I_1(X_+, [H_Q])$ of ideal sheaves of curves in $X_+ = Q \times \mathbb{P}^1$ with Chern character $c = (1, 0, 0, -PD([H_Q]), -1)$ can be identified with the product of $\mathbb{P}^1$ with the moduli space of primitive rational curves in Q (which consists of 2875 rigid curves for a generic $Q \subseteq \mathbb{P}^4$ [15]), i.e.

$$I_1(X_+, [H_Q]) \cong \bigsqcup_{2875} \mathbb{P}^1.$$

Curves in $\bigsqcup_{2875} \mathbb{C}^*$ have well-defined restriction to trivial line bundles on $(Q \times 0) \sqcup (Q \times \infty)$. For curves in $\bigsqcup_{2875} \{0, \infty\} \subseteq I_1(X_+, [H_Q])$, we introduce Li-Wu's expanded pair to define the restriction map. We denote $X_+[1]_0 = \Delta_{-1} \cup X_+ \cup \Delta_1$, where $\Delta_{\pm 1} \cong Q \times \mathbb{P}^1$, and consider the moduli space $I_1(X_+[1]_0, [H_Q])$ of relative ideal sheaves of curves (normal to the divisor $(Q \times 0) \sqcup (Q \times \infty)$). $I_1(X_+[1]_0, [H_Q])$ is the union of $\bigsqcup_{2875} \mathbb{C}^*$ with $\mathbb{C}^*$-equivalence classes of curves inside $\Delta_{\pm 1} \backslash ((Q \times 0) \sqcup (Q \times \infty))$, i.e.

$$I_1(X_+[1]_0, [H_Q]) \cong \bigsqcup_{2875} \mathbb{P}^1.$$

We then have a restriction map

$$I_1(X_+[1]_0, [H_Q]) \to \{\mathcal{O}_{Q \times 0}\} \sqcup \{\mathcal{O}_{Q \times \infty}\},$$

$$I_C \mapsto I_C|_{(Q \times 0) \sqcup (Q \times \infty)} = (\mathcal{O}_{Q \times 0}, \mathcal{O}_{Q \times \infty})$$

to the moduli space of trivial line bundles on $(Q \times 0) \sqcup (Q \times \infty) \subseteq \Delta_{-1} \sqcup \Delta_1$. By direct calculations, for any $I_C \in I_1(X_+[1]_0, [H_Q])$, we have

$$Ext^1_{X_+[1]_0}(I_C, I_C) \cong \mathbb{C}, \quad Ext^3_{X_+[1]_0}(I_C, I_C) \cong \mathbb{C}^2, \quad Ext^i_{X_+[1]_0}(I_C, I_C) = 0, \ i = 2, 4.$$

Conditions in Definition 5.4 are satisfied and the relative DT_4 virtual cycle is the usual fundamental class of the moduli space, i.e. $[I_1(X_+[1]_0, [H_Q])] \in H_2(I_1(X_+[1]_0, [H_Q]), \mathbb{Z})$.

We give a gluing formula of relative DT_4 invariants for the above example.

Example 4.3. In Example 4.2 (ii), we consider $X_0 = X_+ \cup_Y X_-$ and its smoothing $X_t = Q \times \mathbb{T}^2$, where $X_\pm \cong Q \times \mathbb{P}^1$ and $Y = (Q \times 0) \sqcup (Q \times \infty)$. The moduli space $I_1(X_\pm[1]_0, [H_Q])$ of relative ideal sheaves of curves satisfies

$$I_1(X_\pm[1]_0, [H_Q]) \cong \bigsqcup_{2875} \mathbb{P}^1.$$

The relative DT_4 virtual cycle is usual fundamental class of $I_1(X_\pm[1]_0, [H_Q])$. Meanwhile, the moduli space $I_1(X_t, [H_Q])$ of ideal sheaves of curves in $X_t = Q \times \mathbb{T}^2$ with curve class $[H_Q]$ satisfies

$$I_1(X_t, [H_Q]) \cong \bigsqcup_{2875} \mathbb{T}^2,$$

and its DT_4 virtual cycle is the usual fundamental class of $I_1(X_t,[H_Q])$ [13]. Under the homologous relation $X_0 \sim X_t$, we have $\mathbb{P}^1 \cup_{\{0,\infty\}} \mathbb{P}^1 \sim \mathbb{T}^2$, then the gluing formula is expressed by pairing μ-map with these cycles (see Theorem 5.12 for such a formula).

Example 4.4. (Generic quintic in $\mathbb{P}^4$)

We take $X_+ = \mathbb{P}^4$ which contains a smooth generic quintic 3-fold $Y = Q$ as its anti-canonical divisor, and then $H_2(X_+,\mathbb{Z}) \cong H_2(Q,\mathbb{Z}) \cong \mathbb{Z}$. We consider the primitive curve class $[H] \in H_2(X_+,\mathbb{Z})$ and ideal sheaves of curves representing this class have Chern character $c = (1,0,0,-PD([H]),\frac{3}{2})$. We denote their moduli space by $I_{\frac{3}{2}}(X_+,[H])$ ($\cong Gr(2,5)$). The generic quintic Q contains 2875 rigid degree 1 rational curves and $I_{\frac{3}{2}}(X,[H])$ contains a finite subset S with 2875 points. $I_{\frac{3}{2}}(X,[H])\backslash S$ has a well-defined restriction morphism to $Hilb^5(Q)$. To extend the morphism across those 2875 points, we introduce Li-Wu's expanded pair.

We denote $X_+[1]_0 = X_+ \cup \Delta_1$, $Y[1]_0 (\cong Q) \subseteq \Delta_1$, where $\Delta_1 \cong \mathbb{P}(\mathcal{O}_Q \oplus \mathcal{O}_{\mathbb{P}^4}(5)|_Q)$, and consider the moduli space $I_{\frac{3}{2}}(X_+[1]_0,[H])$ of relative ideal sheaves of curves (normal to the divisor Q). Geometrically, it is the blow up of $I_{\frac{3}{2}}(X_+,[H])$ along those 2875 points, i.e.

$$I_{\frac{3}{2}}(X_+[1]_0,[H]) \cong Bl_S(Gr(2,5)),$$

and each exceptional divisor corresponds to a $Hilb^5(\mathbb{P}^1)$ ($\cong \mathbb{P}^5$) for each $\mathbb{P}^1 \subseteq Q$. We then have a restriction morphism

$$I_{\frac{3}{2}}(X_+[1]_0,[H]) \rightarrow Hilb^5(Y[1]_0),$$

$$I_C \mapsto I_C|_{Y[1]_0},$$

which is injective with smooth image. By direct calculations, we have

$$Ext^1_{X_+[1]_0}(I_C,I_C) \cong \mathbb{C}^6, \quad Ext^2_{X_+[1]_0}(I_C,I_C) \cong \mathbb{C}^9, \quad Ext^3_{X_+[1]_0}(I_C,I_C) = 0,$$

and a long exact sequence

$$0 \rightarrow Ext^1_{X_+[1]_0}(I_C,I_C) \rightarrow Ext^1_{Y[1]_0}(I_C|_Y,I_C|_Y) \rightarrow Ext^2_{X_+[1]_0}(I_C,I_C)^* \rightarrow$$

$$\rightarrow Ext^2_{X_+[1]_0}(I_C,I_C) \rightarrow Ext^1_{Y[1]_0}(I_C|_Y,I_C|_Y)^* \rightarrow Ext^1_{X_+[1]_0}(I_C,I_C)^* \rightarrow 0.$$

As $I_C|_Y \in Hilb^5(Y[1]_0)$ is a smooth point with $Ext^1_{Y[1]_0}(I_C|_Y,I_C|_Y) \cong \mathbb{C}^{15}$, conditions in Definition 5.4 are satisfied. The relative DT_4 virtual cycle is the usual fundamental class of the moduli space $I_{\frac{3}{2}}(X_+[1]_0,[H]) \cong Bl_S(Gr(2,5))$.

We adapt Li-Wu's expanded degenerations to torsion sheaves and consider the following extension of Example 4.2.

Example 4.5. (Relative DT_4/DT_3)

Let $X_+ = Y_1 \times \mathbb{P}^1$ which contains $Y = (Y_1 \times 0) \sqcup (Y_1 \times \infty)$ as an anti-canonical divisor, where Y_1 is a compact Calabi-Yau 3-fold. We denote $\mathfrak{M}_c(Y_1)$ to be a Gieseker moduli space of torsion-free semi-stable sheaves on Y_1 with Chern character $c \in H^{even}(Y_1,\mathbb{Q})$ (we assume there is no strictly semi-stable sheaf), and denote $\mathfrak{M}_c(X_+)$ to be the moduli space of sheaves on X_+ which are push-forward of stable sheaves in $\mathfrak{M}_c(Y_1 \times t)$ for some t ($\mathfrak{M}_c(X_+) \cong \mathfrak{M}_c(Y_1) \times \mathbb{P}^1$). Let $\iota : Y_1 \times t \hookrightarrow X_+$ be the natural embedding. For any stable sheaf $\mathcal{F} \in \mathfrak{M}_c(Y_1)$, as in Lemma 6.4 of [13], we have

$$Ext^1_{X_+}(\iota_*\mathcal{F},\iota_*\mathcal{F}) \cong Ext^1_{Y_1}(\mathcal{F},\mathcal{F}) \oplus Ext^0_{Y_1}(\mathcal{F},\mathcal{F}),$$

$$Ext^2_{X_+}(\iota_*\mathcal{F},\iota_*\mathcal{F}) \cong Ext^2_{Y_1}(\mathcal{F},\mathcal{F}) \oplus Ext^2_{Y_1}(\mathcal{F},\mathcal{F})^*. \tag{6}$$

Furthermore, under the above identifications, a Kuranishi map

$$\kappa_{\iota_*\mathcal{F}} : Ext^1_{X_+}(\iota_*\mathcal{F},\iota_*\mathcal{F}) \rightarrow Ext^2_{X_+}(\iota_*\mathcal{F},\iota_*\mathcal{F})$$

satisfies

$$\kappa_{\iota_*\mathcal{F}}(a,b) = (\kappa_{\mathcal{F}}(a),0),$$

for some Kuranishi map $\kappa_{\mathcal{F}} : Ext^1_{Y_1}(\mathcal{F},\mathcal{F}) \rightarrow Ext^2_{Y_1}(\mathcal{F},\mathcal{F})$ of $\mathfrak{M}_c(Y_1)$ at $\mathcal{F}$.

To have a well-defined restriction map, we introduce $X_+[1]_0 = \Delta_{-1} \cup X_+ \cup \Delta_1$, where $\Delta_{\pm 1} \cong Y_1 \times \mathbb{P}^1$, and consider $\mathfrak{M}_c(X_+[1]_0)$ to be the union of $\mathfrak{M}_c(Y_1) \times \mathbb{C}^*$ with the $\mathbb{C}^*$-equivalence classes of $(\mathfrak{M}_c(Y_1 \times 0) \times \mathbb{C}^*) \sqcup (\mathfrak{M}_c(Y_1 \times \infty) \times \mathbb{C}^*)$, i.e.

$$\mathfrak{M}_c(X_+[1]_0) \cong \mathfrak{M}_c(Y_1) \times \mathbb{P}^1.$$

As the supports are disjoint, we have $\mathcal{T}or^{i\geq 1}(\iota_*\mathcal{F},\mathcal{O}_Y) = 0$ which implies a long exact sequence similar to the one in Lemma 5.3, and get a canonical isomorphism $Ext^2_{X_+[1]_0}(\iota_*\mathcal{F},\iota_*\mathcal{F}) \cong$

$Ext^2_{X_+[1]_0}(\iota_*\mathcal{F}, \iota_*\mathcal{F})^*$ as (10). In this specific case, there exists a canonical maximal isotropic subspace $Ext^2_{Y_1}(\mathcal{F}, \mathcal{F}) \subseteq Ext^2_{X_+[1]_0}(\iota_*\mathcal{F}, \iota_*\mathcal{F})$ (6) and $Ext^2_{Y_1}(\mathcal{F}, \mathcal{F})$'s form a sheaf over $\mathfrak{M}_c(X_+[1]_0)$, thus Conjectures 5.8, 5.9 hold. Analogs to Theorem 6.5 [13], the relative DT_4 virtual cycle satisfies

$$[\mathfrak{M}_c^{rel}(X_+[1]_0)]^{vir} = DT_3(\mathfrak{M}_c(Y_1)) \cdot [\mathbb{P}^1] \in H_2(\mathfrak{M}_c(X_+[1]_0), \mathbb{Z}),$$

where $DT_3(\mathfrak{M}_c(Y_1))$ is the Donaldson-Thomas invariant defined by Thomas [53].

5. Appendix on relative DT_4 invariants for ideal sheaves and gluing formulas

5.1. **Li-Wu's good degeneration of Hilbert schemes.** In this subsection, we recall some basic notions and facts of Li-Wu's good degeneration of Hilbert schemes. The precise definitions are left to their papers [39], [56].

The stack of expanded degenerations. We first introduce the stack of expanded degenerations for pairs $(X_\pm, Y)$. We replace a pair (X_+, Y) by expanded pairs of length n, $(X_+[n]_0, Y[n]_0)$, i.e.

$$X_+[n]_0 = X_+ \cup \Delta_1 \cup \cdots \cup \Delta_n,$$

which is a chain of smooth irreducible components intersecting transversally with Δ_i to be the i^{th} copy of $\Delta \triangleq \mathbb{P}(\mathcal{N}_{Y/X_+} \oplus \mathcal{O}_Y)$. Δ is a $\mathbb{P}^1$ bundle over Y with two canonical divisors $Y_\pm$ such that $\mathcal{N}_{Y_+/X_+} \cong \mathcal{N}_{Y/X_+}$ and $\mathcal{N}_{Y_-/X_+} \cong \mathcal{N}^*_{Y/X_+}$. We denote $Y[n]_0 = Y_+$ to be the divisor in the last component Δ_n. In fact, we can consider families of expanded pairs, $(X_+[n], Y[n])$ over affine space $\mathbb{A}^n$ such that over $0 \in \mathbb{A}^n$ it coincides with $(X_+[n]_0, Y[n]_0)$. Then there exists a pair of Artin stacks $(\mathfrak{X}_+, \mathfrak{Y})$ as the direct limit of stack quotients of $(X_+[n], Y[n])$ by certain group actions. The projection of $(X_+[n], Y[n])$ to the affine space $\mathbb{A}^n$ induces a morphism $\mathfrak{Y} \subseteq \mathfrak{X}_+ \to \mathfrak{A}_\diamond$, where $\mathfrak{A}_\diamond$ is the direct limit of some stack quotients of the affine space $\mathbb{A}^{n+1}$.

To formula the gluing formula, we also need to replace the family $\mathcal{X} \to C$ by its expanded degeneration $\mathfrak{X} \to \mathfrak{C}$, where $\mathfrak{X}$ is the direct limit of stack quotients of $X[n]$ and $X[n]$ is a family over $C[n] \triangleq C \times_{\mathbb{A}^1} \mathbb{A}^{n+1}$, $\mathfrak{C} \triangleq C \times_{\mathbb{A}^1} \mathfrak{A}$ and $\mathfrak{A}$ is another stack quotient of the affine space $\mathbb{A}^{n+1}$. A smooth chart of $\mathfrak{X}_0 \triangleq \mathfrak{X} \times_C 0$ is

$$X[n]_0 = X_+ \cup \Delta_1 \cup \cdots \cup \Delta_n \cup X_-,$$

which is a chain of smooth irreducible components intersecting transversally with Δ_i to be the i^{th} copy of $\Delta \triangleq \mathbb{P}(\mathcal{N}_{Y/X_+} \oplus \mathcal{O}_Y) \cong \mathbb{P}(\mathcal{N}_{Y/X_-} \oplus \mathcal{O}_Y)$. We also denote $\Delta_0 = X_+$, $\Delta_{n+1} = X_-$.

If we consider $X[n]_0$ as the gluing of $(X_\pm[n]_0, Y[n]_0)$, we need to specify one of its divisor in some Δ_i. This is called a node-marking and there exists an Artin stack $\mathfrak{X}_0^\dagger$ which is the collection of families in $\mathfrak{X}_0$ with node-markings. One can construct a stack $\mathfrak{C}_0^\dagger$ and an arrow $\mathfrak{C}_0^\dagger \to \mathfrak{C}$ that fits into a Cartesian product

$$\begin{array}{ccc} \mathfrak{X}_0^\dagger & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \mathfrak{C}_0^\dagger & \longrightarrow & \mathfrak{C} \end{array}$$

By Proposition 2.13 [39], there exists a canonical isomorphism $\mathfrak{C}_0^\dagger \cong \mathfrak{A}_\diamond$.

To fix Hilbert polynomials of ideal sheaves over $X[n]_0$ and decompose them into ideal sheaves of fixed Hilbert polynomials on $(X_\pm[n]_0, Y[n]_0)$, we introduce

$$\Lambda_P^{spl} \triangleq \{\delta = (\delta_\pm, \delta_0) \mid \delta_+ + \delta_- - \delta_0 = P\},$$

where $\delta_\pm$, δ_0, P are polynomials in $\mathcal{A} \triangleq \mathcal{A}^* \cup \{0\}$, and $\mathcal{A}^*$ is the set of $\mathbb{Q}$-coefficient polynomials whose leading terms are of the form $a_r \frac{k^r}{r!}$ with $a_r \in \mathbb{Z}_+$.

We define the stack $\mathfrak{X}_0^{\dagger,\delta}$ whose closed points are $(X[n]_0, Y_k, w)$, where w is a function such that

$$w(\Delta_{[0,k-1]}) = \delta_-, \quad w(\Delta_{[k,n+1]}) = \delta_+, \quad w(Y_k) = \delta_0$$

We similarly assign functions $w_\pm$ to $(X_\pm[n]_0, Y[n]_0)$ with

$$w_\pm(\Delta_{[0,n]}) = \delta_\pm, \quad w_\pm(Y[n]_0) = \delta_0$$

and define stacks $\mathfrak{X}_{\pm}^{\delta_\pm,\delta_0}$. Then there exists stacks $\mathfrak{A}_{\diamond}^{\delta_\pm,\delta_0}$ so that we have Cartesian product

$$\begin{tikzcd} \mathfrak{X}_{\pm}^{\delta_\pm,\delta_0} \arrow[r] \arrow[d] & \mathfrak{X}_{\pm} \arrow[d] \\ \mathfrak{A}_{\diamond}^{\delta_\pm,\delta_0} \arrow[r] & \mathfrak{A}_{\diamond} \end{tikzcd}$$

By gluing two components, we obtain the following commutative diagram

$$\begin{tikzcd} \mathfrak{X}_{+}^{\delta_\pm,\delta_0} \times \mathfrak{X}_{-}^{\delta_\pm,\delta_0} \arrow[r] \arrow[d] & \mathfrak{X}_0^{\dagger,\delta} \arrow[d] \\ \mathfrak{A}_{\diamond}^{\delta_+,\delta_0} \times \mathfrak{A}_{\diamond}^{\delta_-,\delta_0} \arrow[r, "\cong"] & \mathfrak{C}_0^{\dagger,\delta} \end{tikzcd}.$$

We denote $\mathfrak{C}_0^{\dagger,P} = \coprod_{\delta\in\Lambda_P^{spl}} \mathfrak{C}_0^{\dagger,\delta}$ and then have a natural morphism $\Phi_\delta : \mathfrak{C}_0^{\dagger,\delta} \to \mathfrak{C}^P$ as the composition of the imbedding $\mathfrak{C}_0^{\dagger,\delta} \to \mathfrak{C}_0^{\dagger,P}$ with forgetful map $\mathfrak{C}_0^{\dagger,P} \to \mathfrak{C}^P$.

Lemma 5.1. *(Li-Wu, Proposition 2.19* [39]*)*
There are canonical line bundles with sections (L_δ, s_δ) on $\mathfrak{C}^P$ indexed by $\delta \in \Lambda_P^{spl}$, such that

(1) let t be the standard coordinate function on $\mathbb{A}^1$ and $\pi : \mathfrak{C}^P \to \mathbb{A}^1$ be the tautological projection, then

$$\bigotimes_{\delta\in\Lambda_P^{spl}} L_\delta \cong \mathcal{O}_{\mathfrak{C}^P}, \qquad \prod_{\delta\in\Lambda_P^{spl}} s_\delta = \pi^* t;$$

(2) Φ_δ factors through $s_\delta^{-1}(0) \subseteq \mathfrak{C}^P$ and there exists an isomorphism $s_\delta^{-1}(0) \cong \mathfrak{C}_0^{\dagger,\delta}$.

This states that $\mathfrak{C}_0^P \subseteq \mathfrak{C}^P$ is a complete intersection substack with $\coprod_{\delta\in\Lambda_P^{spl}} \mathfrak{C}_0^{\dagger,\delta}$ as its normalization.

Moduli stacks of stable ideal sheaves. By Theorem 4.14 [39], there exists a Deligne-Mumford stack $\mathfrak{I}_{\mathfrak{X}/\mathfrak{C}}^P$ which is finite type, separated and proper over C. It is a good degeneration of Hilbert scheme of subschemes of X/C with fixed Hilbert polynomial P in the sense that

$$\mathfrak{I}_{\mathfrak{X}/\mathfrak{C}}^P \times_C t \cong Hilb^P(X_t), \quad t \neq 0$$

and the central fiber $\mathfrak{I}_{\mathfrak{X}_0/\mathfrak{C}_0}^P \triangleq \mathfrak{I}_{\mathfrak{X}/\mathfrak{C}}^P \times_C 0$ has a good obstruction theory. We recall that any closed point of $\mathfrak{I}_{\mathfrak{X}_0/\mathfrak{C}_0}^P$ is an ideal sheaf I_Z in some $X[n]_0$ such that

(1) $\mathcal{O}_Z$ is normal to all $Y_i \subseteq X[n]_0$, i.e. $Tor^1_{\mathcal{O}_{X[n]_0}}(\mathcal{O}_Z, \mathcal{O}_{Y_i}) = 0$;
(2) $Aut_{\mathfrak{X}}(I_Z)$ is finite.
(3) The Hilbert polynomial of $\mathcal{O}_Z$ is P.

We define

$$\mathfrak{I}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}^\delta \triangleq \mathfrak{I}_{\mathfrak{X}/\mathfrak{C}}^P \times_{\mathfrak{C}^P} \mathfrak{C}_0^{\dagger,\delta}.$$

It parameterizes ideal sheaves I_Z's on $X[n]_0$ with a node-marking $Y_k \subseteq X[n]_0$ so that the Hilbert polynomials of $\mathcal{O}_Z$ restricted to $\cup_{i<k}\Delta_i$, to $\cup_{i\geq k}\Delta_i$ and to Y_k are δ_-, δ_+ and δ_0 respectively.

We can similarly define the moduli stack of stable relative ideal sheaves for $\mathfrak{Y} \subseteq \mathfrak{X}_\pm$ with pair Hilbert polynomial $(\delta_\pm, \delta_0)$, denoted by $\mathfrak{I}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}^{\delta_\pm,\delta_0}$, which are finite type, separated and proper Deligne-Mumford stacks (Theorem 4.15 [39]). The relations between $\mathfrak{I}_{\mathfrak{X}/\mathfrak{C}}^P$, $\mathfrak{I}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}^\delta$ and $\mathfrak{I}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}^{\delta_\pm,\delta_0}$ are described as follows.

Lemma 5.2. *(Li-Wu, Theorem 5.27* [39]*)*
(1) There exists natural restriction morphisms $\mathfrak{I}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}^{\delta_\pm,\delta_0} \to Hilb_Y^{\delta_0}$, where $Hilb_Y^{\delta_0}$ is the Hilbert scheme on Y with fixed Hilbert polynomial δ_0, and an isomorphism

$$\mathfrak{I}_{\mathfrak{X}_-/\mathfrak{A}_\diamond}^{\delta_-,\delta_0} \times_{Hilb_Y^{\delta_0}} \mathfrak{I}_{\mathfrak{X}_+/\mathfrak{A}_\diamond}^{\delta_+,\delta_0} \to \mathfrak{I}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}^\delta.$$

(2) Let (L_δ, s_δ) be as in Lemma 5.1 and $\pi_P : \mathfrak{I}_{\mathfrak{X}/\mathfrak{C}}^P \to \mathfrak{C}^P$ be the natural projection. Then

$$\bigotimes_{\delta\in\Lambda_P^{spl}} \pi_P^* L_\delta \cong \mathcal{O}_{\mathfrak{I}_{\mathfrak{X}/\mathfrak{C}}^P}, \qquad \prod_{\delta\in\Lambda_P^{spl}} \pi_P^* s_\delta = \pi_P^*\pi^* t;$$

As closed substacks of $\mathfrak{I}_{\mathfrak{X}/\mathfrak{C}}^P$, we have $\mathfrak{I}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}^\delta \cong (\pi_P^* s_\delta = 0)$.

5.2. **Relative DT_4 virtual cycles.** We study obstruction theories of Deligne-Mumford stacks $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ and $\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}$.

Lemma 5.3. *We take a closed point $[I_Z]\in \mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond}$ with $Z\subseteq X_+[n]_0$, then for $Y=Y[n]_0$, we have a short exact sequence*

$$0\to I_Z\otimes \mathcal{O}_{X_+[n]_0}(-Y)\to I_Z\to I_Z\otimes\mathcal{O}_Y\to 0 \tag{7}$$

and a long exact sequence

$$\cdots\to Ext^1(I_Z,I_Z\otimes\mathcal{O}_{X_+[n]_0}(-Y))\to Ext^1(I_Z,I_Z)\to Ext^1_Y(I_Z\otimes\mathcal{O}_Y,I_Z\otimes\mathcal{O}_Y)\to$$
$$\to Ext^2(I_Z,I_Z\otimes\mathcal{O}_{X_+[n]_0}(-Y))\to Ext^2(I_Z,I_Z)\to Ext^2_Y(I_Z\otimes\mathcal{O}_Y,I_Z\otimes\mathcal{O}_Y)\to$$
$$\to Ext^3(I_Z,I_Z\otimes\mathcal{O}_{X_+[n]_0}(-Y))\to Ext^3(I_Z,I_Z)\to\cdots.$$

Proof. We tensor $0\to I_Z\to\mathcal{O}_{X_+[n]_0}\to\mathcal{O}_Z\to 0$ with $\mathcal{O}_Y$ and get

$$\mathcal{T}or^{i+1}_{\mathcal{O}_{X_+[n]_0}}(\mathcal{O}_Z,\mathcal{O}_Y)\cong\mathcal{T}or^{i}_{\mathcal{O}_{X_+[n]_0}}(I_Z,\mathcal{O}_Y),\quad i\geq 1. \tag{8}$$

Applying tensor product with $\mathcal{O}_Z$ to $0\to\mathcal{O}_{X_+[n]_0}(-Y)\to\mathcal{O}_{X_+[n]_0}\to\mathcal{O}_Y\to 0$, we get

$$\mathcal{T}or^{i\geq 2}_{\mathcal{O}_{X_+[n]_0}}(\mathcal{O}_Z,\mathcal{O}_Y)=0. \tag{9}$$

These ensure that we have the short exact sequence (7) after tensoring $0\to\mathcal{O}_{X_+[n]_0}(-Y)\to\mathcal{O}_{X_+[n]_0}\to\mathcal{O}_Y\to 0$ with I_Z. We then take $Hom(I_Z,\cdot)$ to (7) and are left to show $Ext^*(I_Z,I_Z\otimes\mathcal{O}_Y)\cong Ext^*_Y(I_Z\otimes\mathcal{O}_Y,I_Z\otimes\mathcal{O}_Y)$. We have a spectral sequence

$$H^*(X_+[n]_0,\mathcal{E}xt^*(I_Z,I_Z\otimes\mathcal{O}_Y))\Rightarrow Ext^*(I_Z,I_Z\otimes\mathcal{O}_Y).$$

By Corollary 2.9 [56], we can take a finite length locally free resolution $E^\bullet\to I_Z\to 0$. Then

$$H^*(X_+[n]_0,\mathcal{E}xt^*(I_Z,I_Z\otimes\mathcal{O}_Y))\cong H^*(X_+[n]_0,\mathcal{E}xt^*(\mathcal{O}_{X_+[n]_0},\mathcal{O}_Y)\otimes End(E^\bullet))$$
$$\cong H^*(X_+[n]_0,End(E^\bullet)\otimes\mathcal{O}_Y)\cong H^*(X_+[n]_0,\iota_*End(E^\bullet|_Y))\cong Ext^*_Y(E^\bullet|_Y,E^\bullet|_Y),$$

where $\iota: Y\hookrightarrow X_+[n]_0$ is the closed imbedding. By (8), (9), we have $\mathcal{T}or^{i\geq 1}_{\mathcal{O}_{X_+[n]_0}}(I_Z,\mathcal{O}_Y)=0$ which implies that $E^\bullet|_Y\to I_Z|_Y\to 0$ is still an resolution. Thus $Ext^*_Y(I_Z|_Y,I_Z|_Y)\cong Ext^*(I_Z,I_Z\otimes\mathcal{O}_Y)$. ☐

The above long exact sequence is the ideal sheaf version of the long exact sequence in Lemma 3.3. We consider the following extension of virtual cycles for bundles (**Cases I-III**) to ideal sheaf cases.

Definition 5.4. Let Y be an anti-canonical divisor of a complex projective 4-fold X_+, and

$$r:\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond}\to Hilb^{\delta_0}_Y$$

be Li-Wu's restriction morphism in Lemma 5.2. We assume $\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond}$ is a smooth moduli scheme (all Kuranishi maps vanish).

The relative DT_4 virtual cycle for $\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond}$ is its usual fundamental class provided that any one of the following conditions holds,
(1) r is surjective between smooth moduli spaces, and the obstruction bundle $Ob(\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond})=0$,
(2) r is injective between smooth moduli spaces (at least when restricted to a neighbourhood $U(r(\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond}))$ of $r(\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond})$ in $Hilb^{\delta_0}_Y$), and $rk(Ob(\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond}))=codim(\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond},U(r(\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond})))$.

A modification by twisting $K^{\frac{1}{2}}_{X_\pm}$. In general, extensions of virtual cycles for bundles (**Cases I-III**) to ideal sheaves are not straightforward. We study the extension for **Case I**.

Proposition 5.5. *We take a smooth Calabi-Yau 3-fold Y in complex projective 4-folds $X_\pm$ as their anti-canonical divisors. We assume any $I_C\in Hilb^{\delta_0}(Y)$ satisfies $Ext^1(I_C,I_C)=0$. Then for any closed point of $\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}$, say $[I_{Z_\pm}]$ with $Z_\pm\subseteq X_\pm[n]_0$, we have canonical isomorphisms*

$$Ext^i_{X_\pm[n]_0}(I_{Z_\pm},I_{Z_\pm})_0\cong Ext^{4-i}_{X_\pm[n]_0}(I_{Z_\pm},I_{Z_\pm})_0^*,\ \ i=1,2. \tag{10}$$

Furthermore, under the isomorphism in Lemma 5.2,

$$\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}\cong\mathfrak{I}^{\delta_-,\delta_0}_{\mathfrak{X}_-/\mathfrak{A}_\diamond}\times_{Hilb^{\delta_0}_Y}\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond},$$

where a closed point of $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^{\dagger}/\mathfrak{C}_0^{\dagger}}$ is written as $I_Z = I_{Z_+} \cup I_{Z_-}$, with $Z \subseteq X[n_+ + n_-]_0$, $Z_{\pm} \subseteq X_{\pm}[n_{\pm}]_0$, we have canonical isomorphisms of trace-free extension groups

$$Ext^*_{X[n_+ + n_-]_0}(I_Z, I_Z)_0 \cong Ext^*_{X_+[n_+]_0}(I_{Z_+}, I_{Z_+})_0 \oplus Ext^*_{X_-[n_-]_0}(I_{Z_-}, I_{Z_-})_0,$$

under which the non-degenerate quadratic forms on $Ext^2_{X[n_+ + n_-]_0}(I_Z, I_Z)_0$ and $Ext^2_{X_{\pm}[n]_0}(I_{Z_{\pm}}, I_{Z_{\pm}})$ (10) are preserved.

Proof. We apply the trace-free version of Lemma 5.3 to the case when Y is the Cartier divisor associated with the dualizing sheaf of $X_{\pm}[n]_0$ and get canonical isomorphisms by Serre duality.

We take a closed point $I_Z \in \mathfrak{I}^{\delta}_{\mathfrak{X}_0^{\dagger}/\mathfrak{C}_0^{\dagger}}$ with $Z \subseteq X[n_+ + n_-]_0 = X_+[n_+]_0 \cup_Y X_-[n_-]_0$ and restrict to get $I_{Z_{\pm}} \subseteq \mathcal{O}_{X_{\pm}[n_{\pm}]}$. As [45], [39], we then get an exact triangle

$$RHom_{X[n_+ + n_-]_0}(I_Z, I_Z)_0 \to \bigoplus RHom_{X_{\pm}[n_{\pm}]_0}(I_Z|_{X_{\pm}[n_{\pm}]_0}, I_Z|_{X_{\pm}[n_{\pm}]_0})_0 \to RHom_Y(I_Z|_Y, I_Z|_Y)_0.$$

We take cohomology, use the assumption to deduce $H^*(RHom_Y(I_Z|_Y, I_Z|_Y)_0) = 0$. $\square$

From Proposition 5.5, one may expect to get a (-2)-shifted symplectic structure on $\mathfrak{I}^{\delta_{\pm},\delta_0}_{\mathfrak{X}_{\pm}/\mathfrak{A}_{\diamond}}$ if any $I_C \in Hilb^{\delta_0}(Y)$ satisfies $Ext^1(I_C, I_C) = 0$. Then as Borisov-Joyce did in [7], one expects to use BBJ's type local Darboux charts, partition of unity and homotopical algebra to obtain D-orbifolds associated with $\mathfrak{I}^{\delta_{\pm},\delta_0}_{\mathfrak{X}_{\pm}/\mathfrak{A}_{\diamond}}$ and $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^{\dagger}/\mathfrak{C}_0^{\dagger}}$. In particular, analogs to Theorem 2.6, there should exist homology classes $[\mathfrak{I}^{\delta_{\pm},\delta_0}_{\mathfrak{X}_{\pm}/\mathfrak{A}_{\diamond}}]^{vir} \in H_*(\mathfrak{I}^{\delta_{\pm},\delta_0}_{\mathfrak{X}_{\pm}/\mathfrak{A}_{\diamond}}, \mathbb{Q})$, $[\mathfrak{I}^{\delta}_{\mathfrak{X}_0^{\dagger}/\mathfrak{C}_0^{\dagger}}]^{vir} \in H_*(\mathfrak{I}^{\delta}_{\mathfrak{X}_0^{\dagger}/\mathfrak{C}_0^{\dagger}}, \mathbb{Q})$ if the associated D-orbifolds of $\mathfrak{I}^{\delta_{\pm},\delta_0}_{\mathfrak{X}_{\pm}/\mathfrak{A}_{\diamond}}$ and $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^{\dagger}/\mathfrak{C}_0^{\dagger}}$ are orientable [29].

However, under the assumption: any $I_C \in Hilb^{\delta_0}(Y)$ satisfies $Ext^1(I_C, I_C) = 0$, the non-degenerate quadratic forms on $Ext^2_{X_{\pm}[n]_0}(I_{Z_{\pm}}, I_{Z_{\pm}})$'s (see Proposition 5.5) don't have to glue together, and the D-orbifold associated with $\mathfrak{I}^{\delta_{\pm},\delta_0}_{\mathfrak{X}_{\pm}/\mathfrak{A}_{\diamond}}$ might not exist (see Example 4.1). For gluing, we discuss the case when there exist a square root $K^{\frac{1}{2}}_{X_{\pm}}$ of $K_{X_{\pm}}$ with $K^{\frac{1}{2}}_{X_{\pm}} \otimes K^{\frac{1}{2}}_{X_{\pm}} \cong K_{X_{\pm}}$.

Lemma 5.6. *Let X_+ be a complex projective 4-fold with a square root $K^{\frac{1}{2}}_{X_+}$, Y_i $(i = 1, 2)$ be two smooth zero loci of sections of $K^{-\frac{1}{2}}_{X_+}$ with $K_{Y_i} = 0$ $(\Leftrightarrow \mathcal{N}_{Y_i/X_+} \cong \mathcal{O}_{Y_i})$ and $Y_1 \cap Y_2 = \emptyset$. We take $Y = Y_1 \sqcup Y_2$ which a smooth anti-canonical divisor of X_+. Then for any closed point $[I_Z] \in \mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_{\diamond}}$ with $Z \subseteq X_+[n]_0$ and $Y[n]_0 = Y_1 \sqcup Y_2$, we have a short exact sequence*

$$0 \to I_Z \otimes K^{\frac{1}{2}}_{X_+[n]_0} \to I_Z \to I_Z \otimes \mathcal{O}_{Y_1} \to 0$$

and a long exact sequence

$$0 \to Ext^1_{X_+[n]_0}(I_Z, I_Z \otimes K^{\frac{1}{2}}_{X_+[n]_0})_0 \to Ext^1_{X_+[n]_0}(I_Z, I_Z)_0 \to Ext^1_{Y_1}(I_Z \otimes \mathcal{O}_{Y_1}, I_Z \otimes \mathcal{O}_{Y_1})_0 \to$$

$$\to Ext^2_{X_+[n]_0}(I_Z, I_Z \otimes K^{\frac{1}{2}}_{X_+[n]_0})_0 \to Ext^2_{X_+[n]_0}(I_Z, I_Z)_0 \to Ext^2_{Y_1}(I_Z \otimes \mathcal{O}_{Y_1}, I_Z \otimes \mathcal{O}_{Y_1})_0 \to \cdots,$$

where $K^{\frac{1}{2}}_{X_+[n]_0}$ is a square root of the dualizing sheaf of $X_+[n]_0$.

Furthermore, if any $I_C \in Hilb^{\delta_0}(Y)$ satisfies $Ext^1_Y(I_C, I_C) = 0$, we have canonical isomorphisms

$$\phi_1 : Ext^2_{X_+[n]_0}(I_Z, I_Z \otimes K_{X_+[n]_0})_0 \cong Ext^2_{X_+[n]_0}(I_Z, I_Z \otimes K^{\frac{1}{2}}_{X_+[n]_0})_0,$$

$$\phi_2 : Ext^2_{X_+[n]_0}(I_Z, I_Z \otimes K^{\frac{1}{2}}_{X_+[n]_0})_0 \cong Ext^2_{X_+[n]_0}(I_Z, I_Z)_0,$$

Proof. It is similar to the proof of Lemma 5.3. The isomorphism ϕ_1 is derived by tensoring $0 \to I_Z \otimes K^{\frac{1}{2}}_{X_+[n]_0} \to I_Z \to I_Z \otimes \mathcal{O}_{Y_1} \to 0$ with $K^{\frac{1}{2}}_{X_+[n]_0}$ and taking the long exact sequence. $\square$

The reason of introducing $Ext^2_{X_+[n]_0}(I_Z, I_Z \otimes K^{\frac{1}{2}}_{X_+[n]_0})_0$ is that Serre duality pairing defines a natural non-degenerate quadratic form on it. If $Ext^2_{X_+[n]_0}(I_{Z_+}, I_{Z_+} \otimes K^{\frac{1}{2}}_{X_+[n]_0})_0$'s are glued to be a sheaf over the moduli space, the Serre duality pairing will probably 'glue'. By Proposition 5.5, there is a non-degenerate quadratic form on $Ext^2_{X_+[n]_0}(I_Z, I_Z)_0$. We make a comparison between them.

Proposition 5.7. *Let X_+ be a complex projective 4-fold with a square root $K^{\frac{1}{2}}_{X_+}$, Y_i $(i = 1, 2)$ be two smooth zero loci of sections of $K^{-\frac{1}{2}}_{X_+}$ with $K_{Y_i} = 0$ $(\Leftrightarrow \mathcal{N}_{Y_i/X_+} \cong \mathcal{O}_{Y_i})$ and $Y_1 \cap Y_2 = \emptyset$. We take $Y = Y_1 \sqcup Y_2$ which a smooth anti-canonical divisor of X_+ and assume any $I_C \in Hilb^{\delta_0}(Y)$*

satisfies $Ext^1_Y(I_C, I_C) = 0$. *Then for any closed point* $[I_Z] \in \mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond}$ *with* $Z \subseteq X_+[n]_0$ *and* $Y[n]_0 = Y_1 \sqcup Y_2$, *we have a commutative diagram*

$$\begin{tikzcd}
Ext^2_{X_+[n]_0}(I_Z, I_Z \otimes K_{X_+[n]_0})_0 \arrow[d, "\phi_1"', "\cong"] \arrow[dr, "\phi_3", "\cong"'] & \\
Ext^2_{X_+[n]_0}(I_Z, I_Z \otimes K^{\frac{1}{2}}_{X_+[n]_0})_0 \arrow[r, "\cong", "\phi_2"'] & Ext^2_{X_+[n]_0}(I_Z, I_Z)_0,
\end{tikzcd}$$

where ϕ_1, ϕ_2 *are defined in Lemma 5.6 and* ϕ_3 *is the isomorphism induced from the long exact sequence in Lemma 5.3. Furthermore,* ϕ_2 *is an isometry with respect to the Serre duality pairing on* $Ext^2_{X_+[n]_0}(I_Z, I_Z \otimes K^{\frac{1}{2}}_{X_+[n]_0})_0$ *and the quadratic form on* $Ext^2_{X_+[n]_0}(I_Z, I_Z)_0$ *defined in Proposition 5.5.*

Proof. The commutativity is because isomorphisms ϕ_1, ϕ_2 are pairings with sections in $H^0(X_+[n]_0, K^{-\frac{1}{2}}_{X_+[n]_0})$ corresponding to Cartier divisors Y_1, Y_2, and ϕ_3 is the pairing with a section in $H^0(X_+[n]_0, K^{-1}_{X_+[n]_0})$ corresponding to $Y = Y_1 \sqcup Y_2$. Then it is easy to check ϕ_2 is an isometry. □

Conjecture 5.8. *Let* Y *be a smooth Calabi-Yau 3-fold in complex projective 4-folds* $X_\pm$ *as their anti-canonical divisors. We assume any* $I_C \in Hilb^{\delta_0}(Y)$ *satisfies* $Ext^1(I_C, I_C) = 0$. *Then there exists a D-orbifold associated with Deligne-Mumford stack* $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$, *i.e. they have the same underlying topological structures.*

Furthermore, if canonical bundles of $X_\pm$ *admit square roots* $K^{\frac{1}{2}}_{X_\pm}$, *and there exist* Y_i $(i = 1, 2)$ *which are smooth zero loci of sections of* $K^{-\frac{1}{2}}_{X_\pm}$ *with* $K_{Y_i} = 0$ ($\Leftrightarrow \mathcal{N}_{Y_i/X_\pm} \cong \mathcal{O}_{Y_i}$) *such that* $Y = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \emptyset$, *then there exist D-orbifolds associated with Deligne-Mumford stacks* $\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}$.

Conjecture 5.9. *We take a smooth Calabi-Yau 3-fold* Y *in complex projective 4-folds* $X_\pm$ *as their anti-canonical divisors. We assume any* $I_C \in Hilb^{\delta_0}(Y)$ *satisfies* $Ext^1(I_C, I_C) = 0$ *(we assume* $Hilb^{\delta_0}(Y)$ *consists of one point without loss of generality), then we have*

$$[\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}]^{vir} \in H_*(\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}, \mathbb{Q}), \ [\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}]^{vir} \in H_*(\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}, \mathbb{Q})$$

if there exist orientable D-orbifolds [29] *associated with* $\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}$ *and* $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$. *Furthermore, under the isomorphism*

$$\mathfrak{I}^{\delta_-,\delta_0}_{\mathfrak{X}_-/\mathfrak{A}_\diamond} \times_{Hilb^{\delta_0}_Y} \mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond} \cong \mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$$

in Lemma 5.2, we have an identification of virtual cycles

$$[\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}]^{vir} = [\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond}]^{vir} \times [\mathfrak{I}^{\delta_-,\delta_0}_{\mathfrak{X}_-/\mathfrak{A}_\diamond}]^{vir},$$

if we choose appropriate orientations on D-orbifolds associated with $\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}$ *and* $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$.

5.3. **A conjectural gluing formula.** We state a conjectural gluing formula of DT_4 invariants for a simple degeneration $\mathcal{X} \to C$ of projective CY_4's. We assume $\omega_{\mathcal{X}/C} = 0$ and $X_0 = X_+ \cup_Y X_-$ with Y as an anti-canonical divisor of $X_\pm$.

We first consider the virtual cycle of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$. Because of the triviality of the relative canonical bundle $\omega_{\mathcal{X}/C} = 0$, as Conjecture 5.8, 5.9, there should exist a D-orbifold associated with $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$ and a Borel-Moore homology class $[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{vir} \in H^{BM}_*(\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}, \mathbb{Q})$ if the D-orbifold is orientable [29].

The comparison of $[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{vir}$ **and** $[\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}]^{vir}$. By Lemma 5.2, $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ is the zero loci of a section $\pi_P^* s_\delta$ of a complex line bundle $\pi_P^* L_\delta$ on $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$. Meanwhile, by [39], the obstruction theory of $\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ is the pull-back of the obstruction theory of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$, thus we should have

$$[\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}]^{vir} = c_1(\pi_P^* L_\delta, \pi_P^* s_\delta) \cap [\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{vir}, \tag{11}$$

where $c_1(\pi_P^* L_\delta, \pi_P^* s_\delta) \in H^2(\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}, \mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} - \mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger})$ is the localized first Chern class (Proposition 19.1.2 [22]) and $c_1(\pi_P^* L_\delta, \pi_P^* s_\delta)\cap : H^{BM}_*(\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}) \to H_{*-2}(\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger})$ is the cap product [28].

Summing over all splitting δ of P, we get

$$\sum_{\delta \in \Lambda_P^{spl}} [\mathfrak{I}^{\delta}_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}]^{vir} = c_1(\bigotimes_{\delta \in \Lambda_P^{spl}} \pi_P^* L_\delta, \prod_{\delta \in \Lambda_P^{spl}} \pi_P^* s_\delta) \cap [\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{vir} \tag{12}$$

The comparison of $[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{vir}$ and $[\mathfrak{I}^P_{X_t}]^{vir}$. For $t \neq 0$, we have $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \times_C t \cong \mathfrak{I}^P_{X_t}$. The obstruction theory of $\mathfrak{I}^P_{X_t}$ is the pull back of the obstruction theory of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$. Without loss of generality, we assume $C = \mathbb{A}^1$ and similarly obtain

$$[\mathfrak{I}^P_{X_t}]^{vir} = c_1(\mathcal{O}_{\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}}, \pi_P^* \pi^* t) \cap [\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{vir}, \tag{13}$$

where $[\mathfrak{I}^P_{X_t}]^{vir}$ is the DT_4 virtual cycle mentioned in Theorem 2.6.

Conjecture 5.10. *We take a simple degeneration $\mathcal{X} \to C$ of projective CY_4's with $\omega_{\mathcal{X}/C} = 0$ such that $X_0 = X_+ \cup_Y X_-$ and Y is an anti-canonical divisor of $X_\pm$. Then there exists a D-orbifold associated with $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$, and a Borel-Moore homology class*

$$[\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}]^{vir} \in H^{BM}_*(\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}, \mathbb{Q})$$

if the corresponding D-orbifold is orientable.

Furthermore, equalities (11), (13) hold if we choose appropriate orientations for D-orbifolds associated with $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$, $\mathfrak{I}^\delta_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}$ and $\mathfrak{I}^P_{X_t}$.

To introduce the gluing formula, we make the following definition.

Definition 5.11. Let $\mathcal{X} \to C$ be a simple degeneration of projective CY_4's such that $X_0 = X_+ \cup_Y X_-$ with Y as an anti-canonical divisor of $X_\pm$. P is a polynomial and $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \to C$ is Li-Wu's good degeneration of $Hilb^P(X_t)$, $t \neq 0$. We assume for any $\delta = (\delta_\pm, \delta_0) \in \Lambda^{spl}_P$, any closed point $I_C \in Hilb^{\delta_0}(Y)$ satisfies $Ext^1_Y(I_C, I_C) = 0$. Then the family version DT_4 invariant of $Hilb^P(X_t)$, $t \neq 0$ is a map

$$DT_4(\mathfrak{I}^P_{X_t}) : Sym^*\big(H_*(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]\big) \to \mathbb{Z}$$

such that

$$DT_4(\mathfrak{I}^P_{X_t})((\gamma_1, P_1), (\gamma_2, P_2), ...) = \int_{[\mathfrak{I}^P_{X_t}]^{vir}} \mu(\gamma_1, P_1) \cup \mu(\gamma_2, P_2) \cup ...,$$

where $\gamma_i \in H_*(X, \mathbb{Z})$, $\mu(,)$ is the μ-map defined in Definition 3.4 with respect to the universal sheaf of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$ and we view $\mathfrak{I}^P_{X_t} \hookrightarrow \mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$ as a closed substack to do integration.

We take a Künneth type decomposition of the cohomology class

$$(\mu(\gamma_1, P_1) \cup \mu(\gamma_2, P_2) \cup ...)|_{\mathfrak{I}^\delta_{\mathfrak{X}_0^\dagger/\mathfrak{C}_0^\dagger}} = \sum_i \tau_{+,\delta,i} \boxtimes \tau_{-,\delta,i} \in H^*(\mathfrak{I}^{\delta_-,\delta_0}_{\mathfrak{X}_-/\mathfrak{A}_\diamond} \times \mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond}).$$

Then the relative DT_4 invariant of $\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}$ with respect to $\tau_{\pm,\delta,i}$ is

$$DT_4(\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond})(\tau_{\pm,\delta,i}) = \int_{[\mathfrak{I}^{\delta_\pm,\delta_0}_{\mathfrak{X}_\pm/\mathfrak{A}_\diamond}]^{vir}} \tau_{\pm,\delta,i} \in \mathbb{Q}.$$

We state a gluing formula of DT_4 invariants on Calabi-Yau 4-folds based on previous conjectures.

Theorem 5.12. *Let $\mathcal{X} \to C$ be a simple degeneration of projective CY_4 such that $\omega_{\mathcal{X}/C} = 0$ and $X_0 = X_+ \cup_Y X_-$ with Y as an anti-canonical divisor of $X_\pm$. P is a polynomial and $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}} \to C$ is Li-Wu's good degeneration of $Hilb^P(X_t)$, $t \neq 0$. We assume for any $\delta = (\delta_\pm, \delta_0) \in \Lambda^{spl}_P$, any closed point $I_C \in Hilb^{\delta_0}(Y)$ satisfies $Ext^1_Y(I_C, I_C) = 0$ (without loss of generality, we assume $Hilb^{\delta_0}(Y)$ consists of one point for simplicity), then for $t \neq 0 \in C$,*

$$DT_4(\mathfrak{I}^P_{X_t})((\gamma_1, P_1), (\gamma_2, P_2), ...) = \sum_{\delta \in \Lambda^{spl}_P, i} DT_4(\mathfrak{I}^{\delta_+,\delta_0}_{\mathfrak{X}_+/\mathfrak{A}_\diamond})(\tau_{+,\delta,i}) \cdot DT_4(\mathfrak{I}^{\delta_-,\delta_0}_{\mathfrak{X}_-/\mathfrak{A}_\diamond})(\tau_{-,\delta,i}),$$

where $\gamma_i \in H_*(X, \mathbb{Z})$, $\mu(,)$ is the μ-map defined in Definition 3.4 with respect to the universal sheaf of $\mathfrak{I}^P_{\mathfrak{X}/\mathfrak{C}}$, $(\tau_{\pm,\delta,i})$ is the factor in a Künneth type decomposition as in Definition 5.11.

Proof. By Lemma 5.2, Conjecture 5.9, 5.10 and (12). $\square$

Remark 5.13. By Corollary 2.16 [56], if P is the Hilbert polynomial associated with structure sheaves of points. The condition which says for any $\delta = (\delta_\pm, \delta_0) \in \Lambda^{spl}_P$ and any closed point $I_C \in Hilb^{\delta_0}(Y)$, we have $Ext^1_Y(I_C, I_C) = 0$ is satisfied.

6. Appendix on the orientability of relative DT_4 theory

In this section, we give a coherent description of orientability issues involved in definitions of relative DT_4 virtual cycles (in Section 3) and then give some partial verification for the existence of orientations.

We take a smooth (Calabi-Yau) 3-fold Y in a complex projective 4-fold X as its anti-canonical divisor, and denote $\mathfrak{M}_X$ to be a moduli space of stable bundles on X with fixed Chern classes. Assuming conditions in Theorem 3.1 are satisfied, we obtain a morphism

$$r : \mathfrak{M}_X \to \mathfrak{M}_Y$$

to a Gieseker moduli space of stable sheaves on Y. We denote the determinant line bundle of $\mathfrak{M}_X$ by $\mathcal{L}_X$ with $\mathcal{L}_X|_E \cong det(H^{odd}(X,EndE)) \otimes det(H^{even}(X,EndE)))^{-1}$ (similarly for $\mathcal{L}_Y \to \mathfrak{M}_Y$). In this set-up, there exists a canonical isomorphism

$$\alpha : (\mathcal{L}_{\mathcal{M}_X})^{\otimes 2} \cong r^*\mathcal{L}_{\mathcal{M}_Y}{}^2.$$

Definition 6.1. A *relative orientation* for morphism r consists of a square root $(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}$ of the determinant line bundle $\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}}$ and an isomorphism

$$\theta : \mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{red}} \cong r^*(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}$$

such that $\theta \otimes \theta \cong \alpha$ holds over $\mathcal{M}_X^{red}$ for the isomorphism α.

Proposition 6.2. *The restriction morphism has a relative orientation in **Cases I-III** individually is equivalent to the existence of an orientation in each corresponding case, i.e.*
*(i) the D-manifold associated with $\mathfrak{M}_X$ is orientable in **Case I**;*
*(ii) the self-dual obstruction bundle is orientable in **Case II**;*
*(iii) the self-dual reduced obstruction bundle is orientable in **Case III**.*

Proof. As $\mathfrak{M}_Y$ is smooth in all cases, $\mathcal{L}_Y$ has a canonical square root given by $det(T\mathfrak{M}_Y)$.

In **Case I**, $\mathfrak{M}_Y$ consists of finite number of points. The existence of relative orientations is obviously equivalent to the existence of orientations for the D-manifold associated with $\mathfrak{M}_X$ (see also Theorem 2.7).

In **Case II**, $H^*(X,EndE)$'s and $H^1(Y,EndE|_Y)$'s are locally constant. We abuse notations and use them also to denote the corresponding bundles. By the short exact sequence

$$0 \to H^3(X,EndE)^* \to H^1(X,EndE) \to r^*H^1(Y,EndE|_Y) \to 0$$

in **Case II**, the relative orientability is equivalent to the structure group of the obstruction bundle $H^2(X,EndE)$ can be reduced to $SO(\bullet,\mathbb{C})$, i.e. the self-dual obstruction bundle is orientable.

In **Case III**, the argument is similar as in **Case II**. □

We have the following partial verification of the existence of relative orientations.

Theorem 6.3. *(Weak relative orientability)*
Let Y be a smooth anti-canonical divisor in a projective 4-fold X with $Tor(H_*(X,\mathbb{Z})) = 0$, $E \to X$ be a complex vector bundle with structure group $SU(N)$, where $N \gg 0$. Let $\mathcal{M}_X$ be a coarse moduli scheme of simple holomorphic structures on E, which has a well-defined restriction morphism

$$r : \mathcal{M}_X \to \mathcal{M}_Y,$$

to a proper coarse moduli scheme of simple bundles on Y with fixed Chern classes.

Then there exists a square root $(\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}$ of $\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}}$ such that

$$c_1(\mathcal{L}_{\mathcal{M}_X}|_{\mathcal{M}_X^{red}}) = r^*c_1((\mathcal{L}_{\mathcal{M}_Y}|_{\mathcal{M}_Y^{red}})^{\frac{1}{2}}),$$

where $\mathcal{L}_{\mathcal{M}_X}$ (resp. $\mathcal{L}_{\mathcal{M}_Y}$) is the determinant line bundle of $\mathcal{M}_X$ (resp. $\mathcal{M}_Y$).

Proof. See the proof of Theorem 4.1 [14]. □

Another partial result is given as follows.

Proposition 6.4. *We assume $H^1(\mathcal{M}_X,\mathbb{Z}_2) = 0$. Then relative orientations for restriction morphism $r : \mathcal{M}_X \to \mathcal{M}_Y$ exist.*

Proof. See Proposition 4.6 of [14]. □

[2]See for instance Lemma 4.2 of [14].

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